Dynamical Systems

Evolution Equations

vector field $f: \mathbb{R}^n \to \mathbb{R}^n$

- time continuous systems $\Rightarrow$ ODE $\frac{d}{dt} x = f(x)$
  - $\rightarrow$ solve for some initial conditions
    $x(0) = x_0$ (multicomponent)

state $x \in \mathbb{R}^n$ $\rightsquigarrow$ state space

- discrete time systems : $x_{n+1} = f(x_n)$; $n = 0, 1, 2, \dots$

↳ don't always have explicit : nummerical sol.
↳ interested in "qualitative" behaviour of solutions
  ↳ long scale, e.g. $t \to \infty$ $\rightsquigarrow$ convergence to stationary state?
  ↳ stationary solution
  ↳ chaotic - sensitive to initial conditions
  ↳ regular vs irregular (chaotic) behaviour
    (usually both exist in same state space)

$\rightsquigarrow$ solution is not necessary to figure this out
  $\Rightarrow$ usually depends on $\underline{f} : \mathbb{R}^n \to \mathbb{R}^n$, vector field

Newton : $m\ddot{x} = f(x)$ - 2^{nd} order
laws of motion : $v = \dot{x}$ $\Rightarrow$ $\boxed{\begin{aligned} \dot{x} &= v \\ \dot{v} &= \frac{1}{m} f(x) \end{aligned}}$ } cannot solve 3-body problems

Poincaré : geometric tools to show chaos in solar system
chaos theory (3-body problem)

Lorenz : influential : Deterministic non-periodic flow
meteorologist $\rightsquigarrow$ Lorenz butterfly - strange attractor
attracts solutions
vs. $\Rightarrow$ butterfly effect
↳ sensitive ~~condition~~ dependence on initial conditions

May: population dynamics
discrete: $x_{n+1} = f(x_n)$; $f(x) = ax(1-x)$, $x \in \mathbb{R}$
$\Rightarrow$ chaotic already for 1D (vs. 3D+ in cont.)

First Order Equations

☆ $\boxed{\frac{dx}{dt} = ax}$, $a \in \mathbb{R}$ parameter

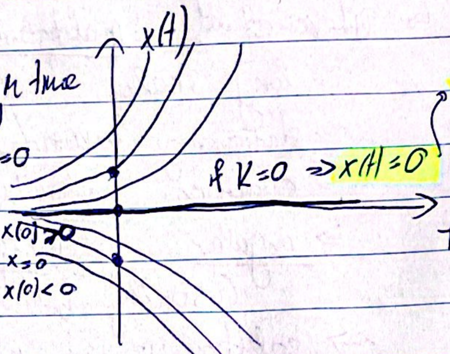
↳ Solution: $x(t) = \underline{Ke^{at}}$, K dependent on initial cond.
namely: $K = x(0)$.

- <u>linear</u> ☺ : LC of sols is also a sol.
- <u>autonomous</u> : RHS is indep. of t

Qualitatively:

<u>$a > 0$</u> : exp increase in time (in magnitude)
or const. if $K=0$

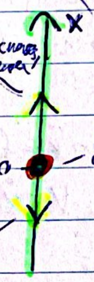
$$\lim_{x \to \infty} x(t) = \begin{cases} +\infty & x(0) > 0 \\ 0 & x = 0 \\ -\infty & x(0) < 0 \end{cases}$$



if $K=0 \Rightarrow x(t)=0$ → stationary state = equilibrium solution

☆ <u>State graph</u> = <u>Phase portrait</u>
~project x-t plane onto x-axis

as t increases x increases

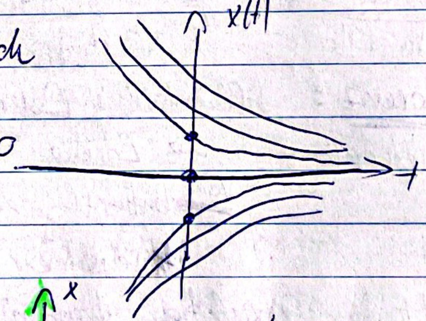


- equilibrium solution: $x=0$
↳ <u>unstable</u> eq. bc arrows pointing away ⇒ source

as t increases, we get lower x

<u>$a < 0$</u> : exp approach to 0.
or const. if $K=0$

$$\lim_{t \to \infty} x(t) = 0$$



⇒ Phase portrait :

increasing t ⇒ decreasing x

- equilibrium solution: $x=0$,
⇒ <u>stable</u> eq. (arrows point towards)
⇒ sink

☆ $\boxed{\frac{dy}{dt} = ay\left(1-\frac{y}{N}\right)}$, non-linear, autonomous ODE

- when $y \ll N$ ⤳ similar as before
  a = per capita growth rate (only for small y)
- N = carrying capacity - when y approaches N, $\frac{dy}{dt} \to 0$

Let $a, N > 0$ (for application purposes)

Two parameters ⤳ $\boxed{x = \frac{y}{N}}$ - new variable

$$\Rightarrow \cancel{N}\frac{dx}{dt} = \cancel{N}ax(1-x)$$

$$\Rightarrow \boxed{\frac{dx}{dt} = ax(1-x)}$$ ⇒ now only one parameter

We can also remove a: $\boxed{\tau = at}$ $\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = a\frac{d}{d\tau}$

$$\frac{dx}{dt} = \cancel{a}\frac{dx}{d\tau} = \cancel{a}x(1-x)$$

$$\Rightarrow \boxed{\frac{dx}{d\tau} = \underbrace{x(1-x)}_{f(x)}}$$ Logistic growth model
- for small x → fast growth
- then level off

- $$\int \frac{dx}{x(1-x)} = \int d\tau = \tau + C$$

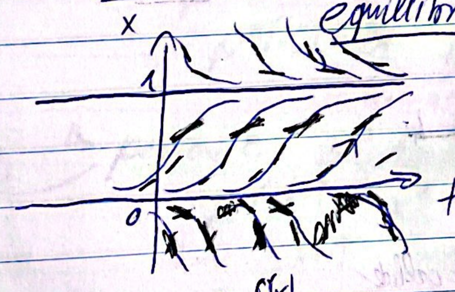
partial fractions $\int \frac{A}{x} + \frac{B}{1-x}\, dx = \int \frac{1}{x} + \frac{1}{1-x}\, dx = \ln|x| + \ln|1-x| = \ln|x(1-x)| = \tau + C$

$$\Rightarrow x(\tau) = \frac{ke^{\tau}}{1+ke^{\tau}}, \quad k = \frac{x(0)}{1-x(0)}$$

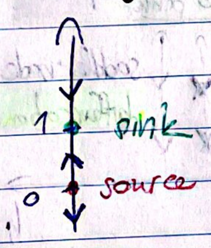
Phase portrait:

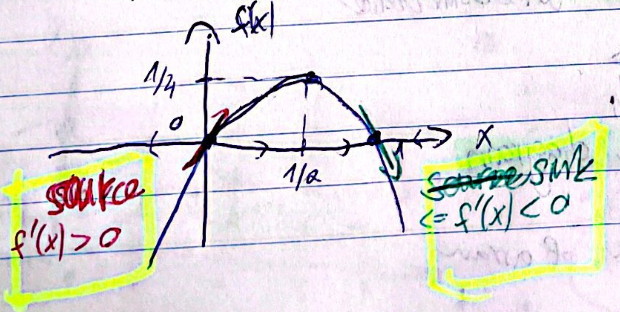
equilibrium points: $\frac{dx}{d\tau} = 0 \Leftrightarrow \underline{x=0} \vee \underline{x=1}$

slope field : ~~…~~
⇒ slope given by $x(1-x)$



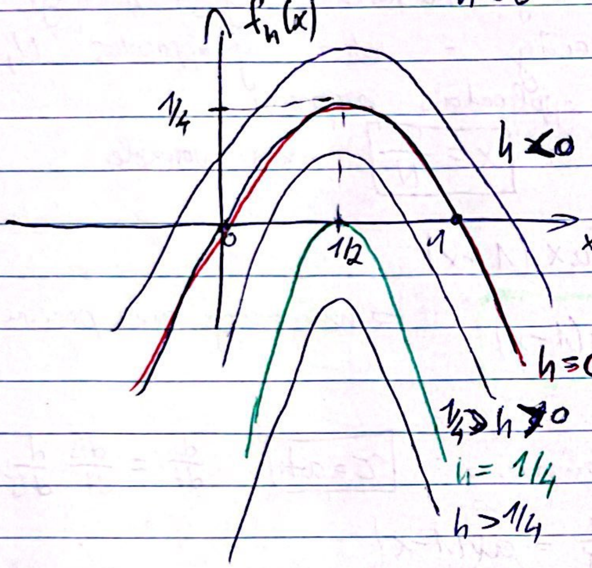
⤳





★ $\boxed{x' = x(1-x) - h =: f_h(x)}$

parameter: $h>0$: remaining population (e.g. fish from pond)
$h<0$: adding at const. rate

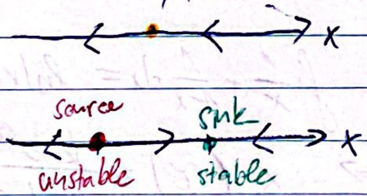


• h-values of equilibria are given by $\boxed{h = x(1-x)}$

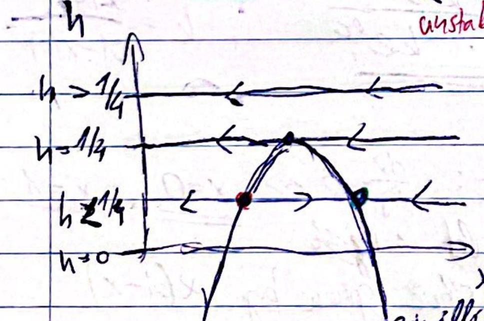
equilibrium only when $h = 1/4$ but sink nor source

$h \geq 1/4$

$1/4 > h$ ~~>0~~

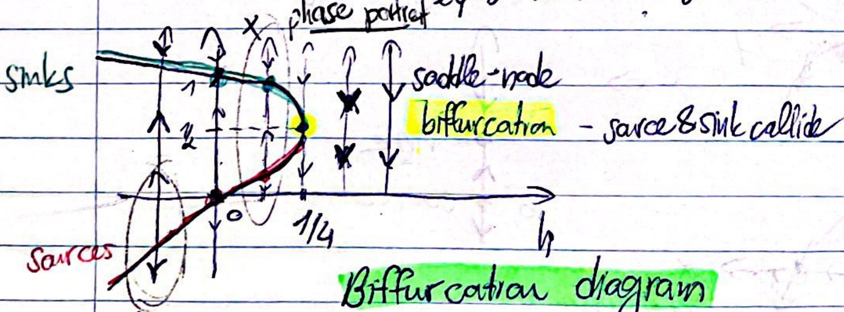


Phase portrait



equilibria ~~for given h~~

phase portrait



Biffurcation diagram

when crossing the line, switch direction of arrow

☆ $\boxed{x' = f(x) = a\,x(1-x) - h\,(1+\sin(2\pi t))} = f(t,x) = f(t+n,x)$, $n \in \mathbb{Z}$.

↳ periodic with period 1

↳ periodicly adding/removing fish

↳ we have rescaled time such that RHS is periodic with period 1

⇒ no more freedom to remove a

↳ periodic as well as non-periodic solutions

(?) ~~periodic solutions (with period 1)~~ ?

→ enough to solve an interval $t \in [0,1]$ (period), ~~rest follows from periodicity~~

since: if $x_1(t)$ is solution on $0 \le t \le 1$ with $x_1(0) = x_0$, and $\exists\, x_2(t)$ sol. on $0 \le t \le 1$ with $x_2(0) = x_1(1)$,

then $x(t) = \begin{cases} x_1(t), & 0 \le t \le 1 \\ x_2(t-1), & 1 \le t \le 2 \end{cases}$ is a solution on $0 \le t \le 2$

Suppose for each x_0, we know the solution $x(t)$ with $x(0) = x_0$. Define map:

$$\boxed{p: x_0 \mapsto x(1)} = \text{Poincaré map}$$

/ stroboscopic / time one map

↳ only "looking" at a value of the solutions at integer times

$p(x_0) = x(1)$, where $x(t)$ is solution with $x(0) = x_0$.

$p(p(x_0)) = p(x(1)) = x(2)$

⇒ !

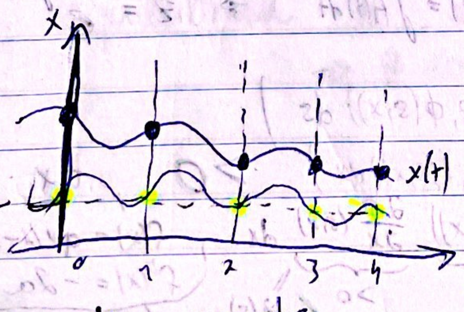
$p^n(x_0) = \underbrace{p(\dots p(x_0))}_{n\text{-times}} = x(n)$, $n \in \mathbb{N}$.

↳ periodic solutions are the fixed points of Poincaré map:

$p(x^*) = x^*$

with same period as RHS (now 1)

then $x(t)$ with $x(0) = x^*$ is a solution with $T = 1$.

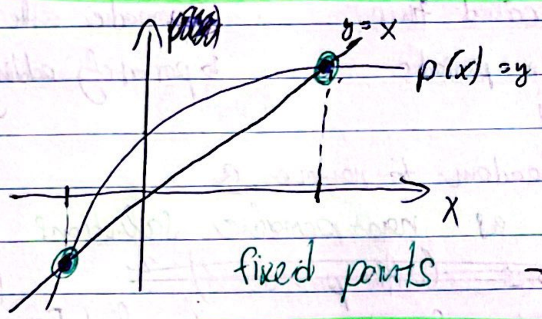


↳ Poincaré map easier to solve because

- discrete (vs. continuous ODE)
- ODE in 2D space (or n+1 space) vs. only 1D space (n-space) of p

→ $y = t$, $x' = f(y,x)$, $y' = 1$

• fundamental concept: describe an ODE in terms of a map on a state space which has one dimension less.



fixed points – where? how many? EXAMINE derivatives of $p(x)$

• notation: $\Phi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ = flow

$(t, x_0) \longmapsto x(t)$

$\quad \overset{\shortparallel}{x(0)}$

$\Rightarrow x(t) = \phi(t, x_0) = \phi_t(x_0) = \phi^t(x_0)$

$\Rightarrow p(x_0) = x(1) = \phi(1, x_0)$

$\Rightarrow p^n(x_0) = \phi(n, x_0)$

$$\Rightarrow \boxed{\phi(t, x_0) = x_0 + \int_0^t f(s, \phi(s, x_0))\, ds} = x(t)$$

$x(0)$ ↳ different way of writing the sol. to ODE, namely: $x(0) = x_0 + 0 = x_0$ ✓

$\frac{dx}{dt} = 0 + f(t, \phi(t, x_0)) = f(t, x)$ ✓

↳ derivatives of p:

$$\frac{dp(x)}{dx_0} = \frac{\partial \phi}{\partial x_0}(1, x_0) = 1 + \int_0^t \frac{\partial f}{\partial x}(s,x) \cdot \frac{\partial \phi}{\partial x}(s,x)\, ds$$

let $z(t) = \frac{\partial \phi(t,x)}{\partial x}$, so $z(0) = 1$ $\quad d/dt$

$$z'(t) = \underbrace{\frac{\partial f}{\partial x}(t, \phi(t,x))}_{A(t)} \underset{\shortparallel\ \frac{\partial \phi}{\partial x}}{z(t)} \quad = \text{variational equation}$$

$z' = \frac{dz}{dt} = A(t)\, z$

$\int_1 \frac{dz}{z} = \int_0 A(t)\, dt \Rightarrow \ln(z) = \int A(t)\, dt \Rightarrow z = e^{\int A(t)\, dt}$

$$\Rightarrow z = \exp\left(\int_0^t \frac{\partial f}{\partial x}(s, \phi(s,x))\, ds \right)$$

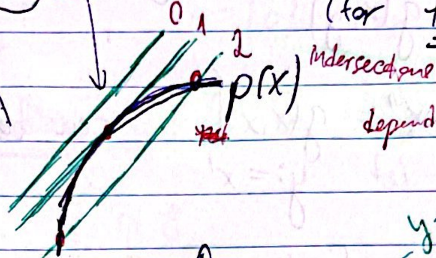
$p'(x_0) = z(1) = \exp\left(\int_0^1 \frac{\partial f}{\partial x}(s, \phi(s,x))\, dx \right) > 0, \quad \forall x \Rightarrow$ monotonic increase

$$p''(x) = p''(x_0) = \underbrace{p'(x)}_{>0} \cdot \int_0^1 \underbrace{\frac{\partial^2 f}{\partial x^2}(s, \phi(s,x))}_{-2a < 0} \underbrace{\frac{\partial \phi}{\partial x}(s,x)}_{>0,\ \shortparallel\ z(s) = \exp(\,) > 0}\, dx$$

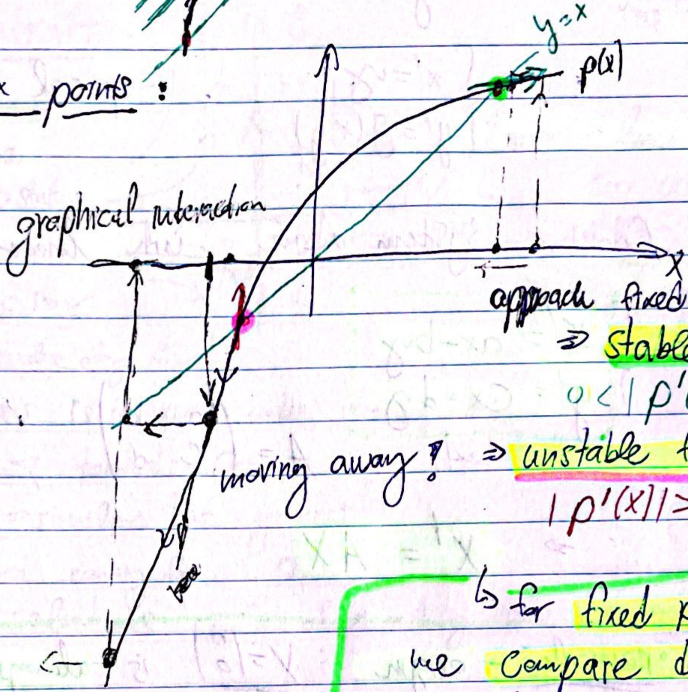
for our example $\left\{ \underbrace{\qquad\qquad}_{<0} \right.$, $f(x) = ax(1-x) +$ time-dep., $f''(x) = -2a$

$\Rightarrow$ $\boxed{\begin{array}{ll} p'(x) > 0 & \forall x \\ p''(x) < 0 & \forall x \end{array}}$ $\Leftarrow$ for all ODE's, consequence of uniqueness of solutions of ODEs

$\Rightarrow$ p has at most 2 fix points (for this ODE)



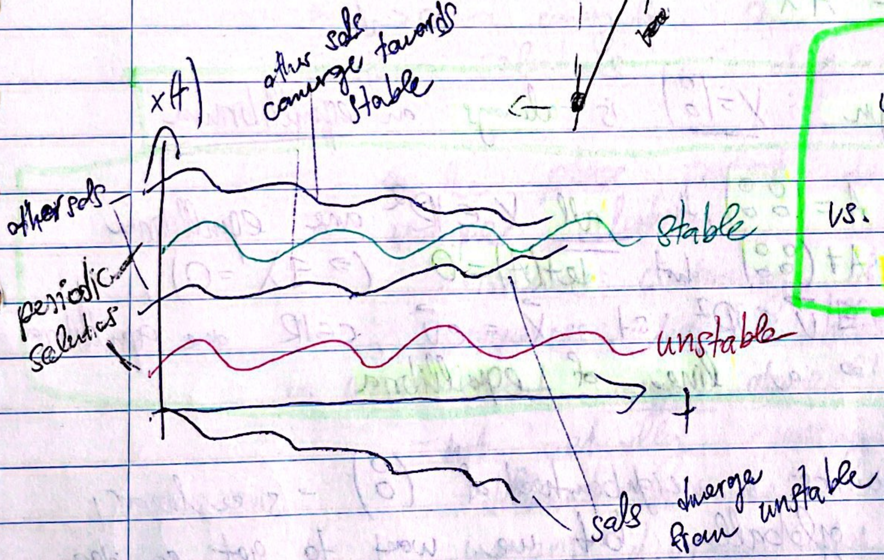
Suppose 2 fix points:



$\Rightarrow$ stable fixed point $0 < |p'(x)| < 1$

$\Rightarrow$ unstable fixed point $|p'(x)| > 1$

$\hookrightarrow$ for fixed points of maps, we compare derivatives to 1 to determine stability vs. comparison to 0 for equilibria of ODEs



every equillibrium is a fixed point
$\Rightarrow$ # equillibria $\leq$ # fixed points

2-dim. state spaces = Planar systems

$$\boxed{\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}}$$ system of 1^{st} order equations

$x'' = g(x,x')$ can be transformed into a system of 1^{st} order:

Newton's law — Let $y := x'$

$\Rightarrow \begin{cases} x' = y \\ y' = g(x,y) \end{cases}$:- special case of system above

Consider system above, with linear f, g:

$$\boxed{\begin{cases} x' = ax+by \\ y' = cx+dy \end{cases}}$$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow X' = AX$

Equilibria

• remark: – origin : $X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is always an equilibrium

– if $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then all $X \in \mathbb{R}^2$ are equilibria

– if $A \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ but $\det(A) = 0$ ($\Rightarrow \exists \lambda = 0$), then $\exists \vec{v} \in \mathbb{R}^2$ s.t. $\vec{X} = c\vec{v}$, $c \in \mathbb{R}$ are equilibria

$\Rightarrow$ a line of equilibria

Now: study dynamics in neighbourhood of $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ – since linear, actually globally but we want to get an idea of stability at origin

First: general sol of $\vec{X}' = A\vec{X}$?

$\hookrightarrow$ study evals and evects

$\lambda_{1,2}$ $\quad\quad$ $\hookrightarrow \vec{v}_{1,2}$

$\vec{x}' = AX$

1) $\lambda_1 \neq \lambda_2$, $\lambda_{1,2} \in \mathbb{R}$; BÚNO: $\lambda_1 < \lambda_2$

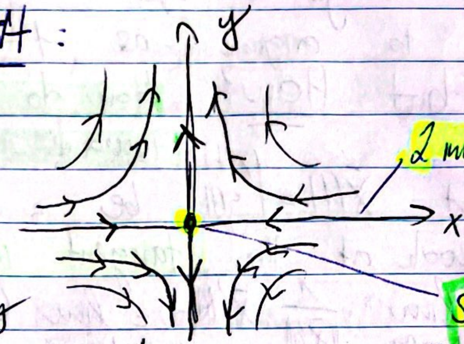
☆ $A = \begin{pmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{pmatrix}$; $v_1 = \begin{pmatrix}1 \\ 0\end{pmatrix}$; $v_2 = \begin{pmatrix}0 \\ 1\end{pmatrix}$

↳ General solution: $\bar{x}(t) = ae^{\lambda_1 t} v_1 + be^{\lambda_2 t} v_2$

with $a, b \in \mathbb{R}$ determined by initial conditions

Phase portrait:

(a) $\lambda_1 < 0 < \lambda_2$



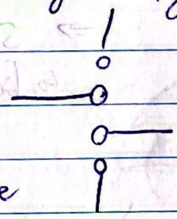
2 invariant lines* through the origin (the saddle)
↳ stable & unstable curves of the saddle

saddle equilibrium

↳ if $b=0$ ⇒ start on x-axis ⇒
⇒ we always stay there (evolve along there)
= x-axis is invariant under the dynamic
↳ $\lambda_1 < 0$ ⇒ towards 0
↳ if $a = 0$ ⇒ similar but y-axis
↳ $\lambda_2 > 0$ ⇒ away from 0
↳ other solutions LC of x and y-axis ⇒ hyperbolas

Lines on the phase portrait do not cross if unique solutions
(existence: continuity of f, g
uniqueness: less than differentiability of f, g needed)

↳ what about origin above?!
- not actually crossing:

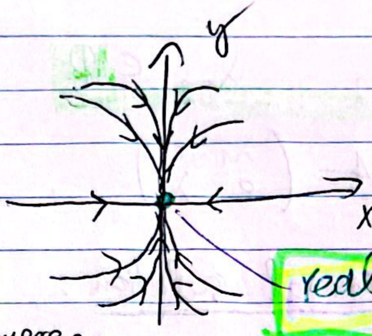


→ different branches
↳ x-axis: 2 branches which take ∞ long to reach origin
↳ y-axis: moving away, if going backwards in time, ∞ time to go back to origin
↳ origin: stay there forever

Generalisation

* 2 invariant lines through the origin are always along the directions of the eigenvectors $\bar{v}_1, \bar{v}_2$

(b) $\boxed{\lambda_1 < \lambda_2 < 0}$



- x, y-axis invariant lines but now along both approach

$\Rightarrow$ LC: always converge to origin as $t \to \infty$

real sink equilibrium

but How? How do solutions converge towards the origin?

Let $\vec{X}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ be a solution.

Look at the tangent vector: $\vec{x}'(t)$

$$\lim_{t\to\infty} \underbrace{\frac{1}{\|\vec{x}'(t)\|}}_{\text{we want a unit vector}} \vec{x}'(t) = \lim_{t\to\infty} \frac{(x'(t), y'(t))^T}{\|\vec{x}'\|}$$

$$= \lim_{t\to\infty} \frac{1}{\|\vec{x}'\|} \begin{pmatrix} a\lambda_1 e^{\lambda_1 t} v_{1x} + b\lambda_2 e^{\lambda_2 t} v_{2x} \\ a\lambda_1 e^{\lambda_1 t} v_{1y} + b\lambda_2 e^{\lambda_2 t} v_{2y} \end{pmatrix}$$

$$= \lim_{t\to\infty} \frac{e^{\lambda_2 t}}{\|\vec{x}'\|} \begin{pmatrix} a\lambda_1 e^{(\lambda_1-\lambda_2)t} v_{1x} + b\lambda_2 v_{2x} \\ a\lambda_1 e^{(\lambda_1-\lambda_2)t} v_{1y} + b\lambda_2 v_{2y} \end{pmatrix}$$

$\|(\ )\|$

$\lambda_1 < \lambda_2 \Rightarrow e^{(\lambda_1-\lambda_2)t} \xrightarrow[t\to\infty]{} 0$

$$= \frac{1}{\|\ \|} \begin{pmatrix} b_2 \lambda_2 v_{2x} \\ b_2 \lambda_2 v_{2y} \end{pmatrix} = \frac{b_2 \lambda_2 \vec{V}_2}{\|\ \ \|} = \frac{\vec{V}_2}{\|\vec{V}_2\|}$$

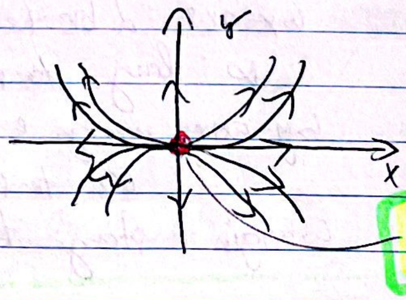
because $\lambda_2 > \lambda_1$

$\Rightarrow$ solutions approach origin along $\vec{V}_2$ line

↳ because $|\lambda_2| < |\lambda_1| \Rightarrow e^{\lambda_1 t}$ goes to 0 faster

$\Rightarrow$ at large t, $e^{\lambda_2 t}$ dominates

(c) $\boxed{\lambda_2 > \lambda_1 > 0}$



- $> 0 \Rightarrow$ increasing, moving away from origin

real source equil.

- in which direction move away? same as above but reverse time and swap x, y because they diverge along lines of eval smaller in magnitude ($|\lambda_2| > |\lambda_1| \Rightarrow \lambda_2$ diverges away too fast)

2) $(\lambda_1 \neq \lambda_2)$ $\lambda_{1,2} \in \mathbb{C}$

Because A is a real matrix, $\lambda_2 = \lambda_1^*$ $\Rightarrow \lambda_1 \neq \lambda_2$ because $\lambda_{1,2} \notin \mathbb{R}$

Set $\lambda := \lambda_1 = \alpha + i\beta$,

$\lambda^* = \lambda_2$

$\alpha = \mathrm{Re}(\lambda)$

$\beta = \mathrm{Im}(\lambda)$

☆ $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $\Rightarrow$ eigenvector: $\vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\Rightarrow \vec{V}(t) = e^{\lambda t} v = e^{(\alpha + i\beta)t} v$ = complex solution

$\Rightarrow$ take LC of Real and imaginary parts

$\mathrm{Re}(\vec{v}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\mathrm{Im}(\vec{v}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\Rightarrow$ general real solution: $x(t) = a\,\mathrm{Re}(v(t)) + b\,\mathrm{Im}(v(t))$

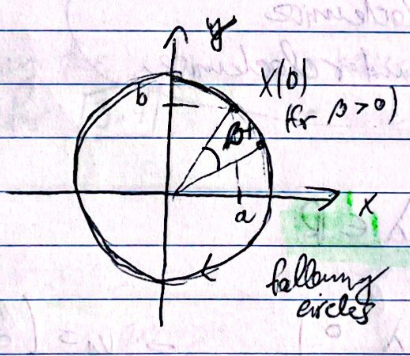
$\downarrow e^{i\beta} = \cos(\beta t) + i\sin(\beta t)$, then decompose into Re & Im

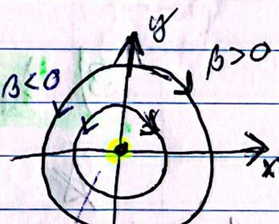
$\Rightarrow x(t) = e^{\alpha t}\left[a\left(\cos(\beta t)\,\mathrm{Re}(v) - \sin(\beta t)\,\mathrm{Im}(v)\right) + b\left(\sin(\beta t)\,\mathrm{Re}(v) + \cos(\beta t)\,\mathrm{Im}(v)\right)\right]$

In our example:

$x(t) = e^{\alpha t} \underbrace{\begin{pmatrix} \cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t) \end{pmatrix}}_{\text{Rotation about the origin by } -\beta t \text{ counterclockwise}} \begin{pmatrix} a \\ b \end{pmatrix}$, $a, b \in \mathbb{R}$ determined by initial conditions

(a) $\alpha = 0$



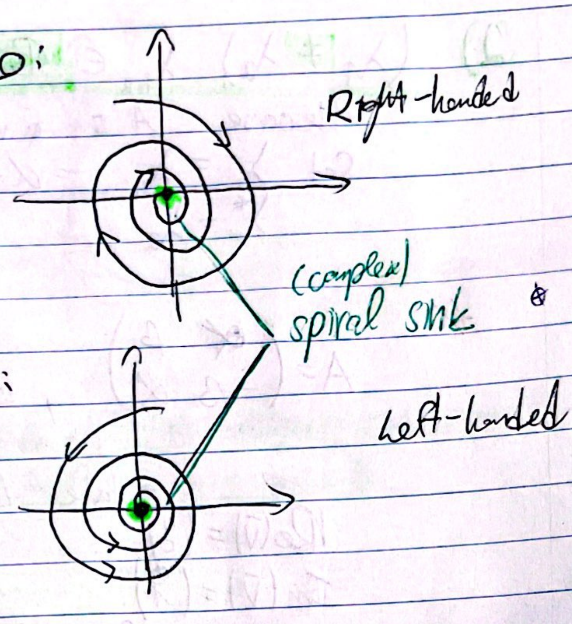


$\Rightarrow$ Phase portrait = concentric circles

center equilibrium

ⓑ $\alpha < 0$:

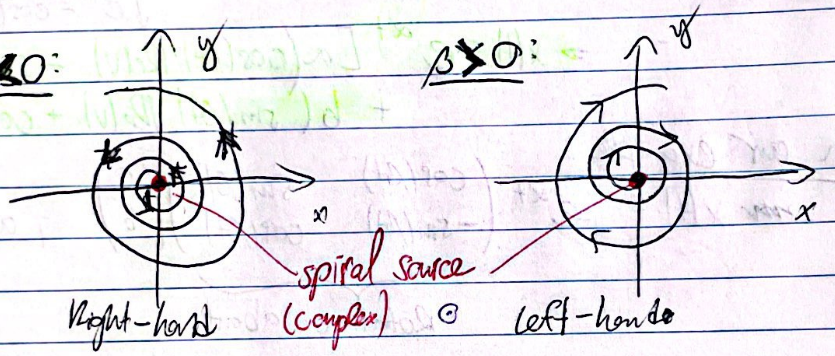
$\beta > 0$:



$\beta < 0$:

ⓒ $\alpha > 0$:

$\beta > 0$: $\beta < 0$:



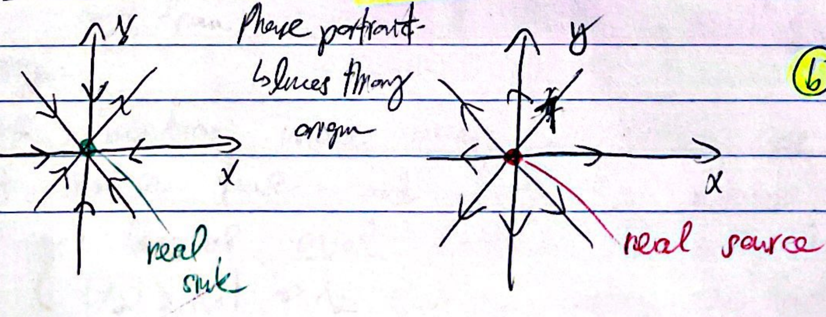
$\boxed{\beta > 0}$	clockwise
$\boxed{\beta < 0}$	counterclockwise

3) $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R}$

2 evectors $\quad A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

general solution: $\bar{x}(t) = e^{\lambda t}(a v_1 + b v_2)$, $a, b \in \mathbb{R}$

ⓐ $\lambda < 0$ ⓑ $\lambda > 0$



☆ $A = \begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ ← Jordan-normal form of 1dimensional eigenspace

1 evector

eigenspace of λ is spanned by $\begin{pmatrix}1\\0\end{pmatrix}$

↳ only 1 eigenvector

general solution: $x(t) = ae^{\lambda t}\begin{pmatrix}1\\0\end{pmatrix} + be^{\lambda t}\begin{pmatrix}t\\1\end{pmatrix}$

$x' = \lambda x + y$

$y' = \lambda y \Rightarrow y(t) = \underline{be^{\lambda t}}$

$x' = \lambda x + be^{\lambda t}$

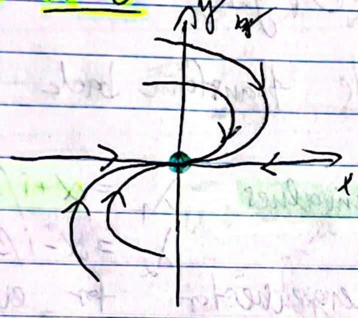
$\Rightarrow x(t) = x_{homo}(t) + x_p(t)$

$\underline{ae^{\lambda t}}$

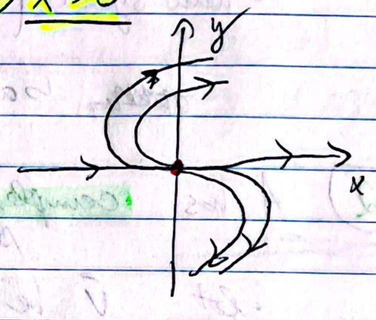
can't take $e^{\lambda t}$ because that's already in homogeneous, so take $\underline{te^{\lambda t}}$

Phase-portrait: exp will again dominate over t

(a) $\lambda < 0$:



(b) $\lambda > 0$:



prove by checking the unit tangent vector $\boxed{3.7}$

- reverse direction of arrow
- take x to $-x$ $\begin{pmatrix}\lambda \to -\lambda\\ 1 \to -1\\ t \to -t\end{pmatrix}$

Canonical forms

• matrices from examples:

real 2dim eigenspace — $\begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$, complex — $\begin{pmatrix}\alpha & +\beta\\ -\beta & \alpha\end{pmatrix}$, $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ — real 1dim eigenspace

are the canonical forms for the corresponding disposition of eigenvalues in the complex plane

① A has real, distinct eigenvalues λ, μ

- $\vec{v}_1, \vec{v}_2$ are eigenvectors
- let $T = (\vec{v}_1 \ \vec{v}_2) = \begin{pmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{pmatrix}$

$$\boxed{T\begin{pmatrix}1\\0\end{pmatrix} = \vec{v}_1, \quad T\begin{pmatrix}0\\1\end{pmatrix} = \vec{v}_2}$$

$$\Rightarrow T^{-1}AT = D = \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$$

$$D\begin{pmatrix}1\\0\end{pmatrix} = T^{-1}AT\begin{pmatrix}1\\0\end{pmatrix} = T^{-1}A\vec{v}_1 = T^{-1}\lambda\vec{v}_1 = \lambda(T^{-1}\vec{v}_1) = \lambda\begin{pmatrix}1\\0\end{pmatrix}$$

$$D\begin{pmatrix}0\\1\end{pmatrix} = T^{-1}AT\begin{pmatrix}0\\1\end{pmatrix} = T^{-1}A\vec{v}_2 = T^{-1}\mu\vec{v}_2 = \mu(T^{-1}\vec{v}_2) = \mu\begin{pmatrix}0\\1\end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix}x_{old}\\y_{old}\end{pmatrix} = A\begin{pmatrix}x_{old}\\y_{old}\end{pmatrix} \quad \& \quad \boxed{\begin{pmatrix}x_{old}\\y_{old}\end{pmatrix} = T\begin{pmatrix}x_{new}\\y_{new}\end{pmatrix}}$$

$$\frac{d}{dt}\begin{pmatrix}x_{new}\\y_{new}\end{pmatrix} = \frac{d}{dt}T^{-1}\begin{pmatrix}x_{old}\\y_{old}\end{pmatrix} = T^{-1}A\begin{pmatrix}x_{old}\\y_{old}\end{pmatrix} = \underbrace{T^{-1}AT}_{D}\begin{pmatrix}x_{new}\\y_{new}\end{pmatrix}$$

$\Rightarrow$ new system $\begin{pmatrix}x_{new}\\y_{new}\end{pmatrix}' = D\begin{pmatrix}x_{new}\\y_{new}\end{pmatrix}$

~~solve~~, ↳ afterwards transform back

② A has complex eigenvalues $\beta \neq 0$ $\quad \lambda_1 = \alpha + i\beta = \lambda, \ \alpha, \beta \in \mathbb{R}$
$\lambda_2 = \alpha - i\beta$

- let $\vec{v}$ be an eigenvector for eval λ
$\Rightarrow \vec{v} \in \mathbb{C}^2$ because $A \in \mathbb{R}^{2\times 2}$

$\Rightarrow$ construct $\vec{v}_1 = \mathrm{Re}(\vec{v})$, $\vec{v}_2 = \mathrm{Im}(\vec{v})$ } always lin. indep.

Proof: by contradiction.
Suppose $\vec{v}_1 = c\vec{v}_2$, $c \in \mathbb{R}$

$$\Rightarrow A(\vec{v}) = \lambda\vec{v}$$

$$(c+i)A\vec{v}_2 = A(\vec{v}_1 + i\vec{v}_2) = (\alpha + i\beta)\vec{v} = (\alpha + i\beta)(c+i)\vec{v}_2$$

$\vec{v}_1, \vec{v}_2 \in \mathbb{R}^2$: $\underbrace{A\vec{v}_2}_{\in\mathbb{R}^2} = \underbrace{(\alpha + i\beta)\vec{v}_2}_{\notin\mathbb{R}^2} \Rightarrow$ contradiction

$$A\vec{v} = A\vec{v}_1 + iA\vec{v}_2 = (\alpha + i\beta)\vec{v} = (\alpha + i\beta)(\vec{v}_1 + i\vec{v}_2) = \alpha\vec{v}_1 - \beta\vec{v}_2 + i(\beta\vec{v}_1 + \alpha\vec{v}_2)$$

$\Rightarrow A\vec{v}_1 = \alpha\vec{v}_1 - \beta\vec{v}_2$

$A\vec{v}_2 = \beta\vec{v}_1 + \alpha\vec{v}_2$

$\Rightarrow$ Let $T = (\vec{v}_1 \ \vec{v}_2) \Rightarrow T^{-1}AT = \begin{pmatrix}\alpha & +\beta\\ -\beta & \alpha\end{pmatrix}$

③ A has ~~repeat~~ (real) repeated eigenvalues $\lambda_1 = \lambda_2 = \lambda$

linearly independent eigenvectors - dimension of eigenspace = 2
↳ see case ①: $\boxed{T = (\vec{v}_1 \ \vec{v}_2) \Rightarrow D = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}}$

linearly dependent eigenvectors - dimension of eigenspace = 1

- let $\vec{v}$ be an eigenvector.
- choose arbitrary vector $\vec{w}$, linearly indep. from $\vec{v}$
- because $\vec{v}, \vec{w}$ span $\mathbb{R}^2$

$$\Rightarrow A\vec{w} = \mu \vec{v} + \nu \vec{w}$$

↳ in fact: $\nu = \lambda \Rightarrow A\vec{w} = \mu\vec{v} + \lambda\vec{w}$

↳ **Proof**: by contradiction. Suppose $\nu \neq \lambda \Leftrightarrow \nu - \lambda \neq 0$

$$A\left(\vec{w} + \frac{\mu}{\nu-\lambda}\vec{v}\right) = \mu\vec{v} + \nu\vec{w} + \frac{\mu}{\nu-\lambda}\lambda\vec{v} =$$
$$= \nu\vec{w} + \frac{\mu}{\nu-\lambda}(\nu - \lambda + \lambda)\vec{v} =$$
$$= \nu\left(\vec{w} + \frac{\mu}{\nu-\lambda}\vec{v}\right)$$

$\Rightarrow \nu$ is an eigenvalue of A $\neq \lambda$

$\Rightarrow$ contradiction, A has only 1 eigenvalue $\square$

- set $\vec{u} := \frac{1}{\mu}\vec{w}$

$\Rightarrow A\vec{u} = \vec{v} + \lambda\vec{u}$ & ~~$A\vec{v} = \lambda\vec{v} + 0\vec{u}$~~

$A\vec{v} = \lambda\vec{v} + 0\vec{u}$

Let $T = (\vec{v} \ \vec{u}) \Rightarrow T^{-1}AT = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

↓

$\boxed{\vec{u} \text{ is an eigenvector of rank } 2 \text{ (generalized evect)} \Rightarrow (A-\lambda I)^2 \vec{u} = 0 \text{ but } (A-\lambda I)\vec{u} \neq 0}$

Proof: $(A-\lambda I)(A-\lambda I)\vec{u} = (A-\lambda I)(v + \lambda u - \lambda u) =$
$= (A-\lambda I)\vec{v} = \vec{0}$ $\square$

② How to solve a linear system?

$\vec{X}' = A\vec{X}$

① find eigenvalues of A: λ_1, λ_2

② find eigenvector(s) $v / v_1, v_2$

③

if $\lambda_1 \neq \lambda_2 \in \mathbb{R}$: general solution is $X(t) = a e^{\lambda_1 t} v_1 + b e^{\lambda_2 t} v_2$

$a, b \in \mathbb{R}$

if $\lambda_1 = \lambda_2$ ($\Rightarrow \in \mathbb{R}$): — 2-dim eigenspace: just with $\lambda_1 = \lambda_2$

— 1-dim eigenspace:

- choose $\vec{u}$ such that $A\vec{u} = \vec{v} + \lambda \vec{u}$ (eigenvector $\vec{v}$, rank-2 evect $\vec{u}$)

① choose arbitrary lin indep. $\vec{w}$
② apply $A\vec{w} = \mu\vec{v} + \lambda\vec{w}$
③ let $\vec{u} = \frac{1}{\mu}\vec{w}$

$\Rightarrow$ general solution: $X(t) = a e^{\lambda t}\vec{v} + b e^{\lambda t}(t\vec{v} + \vec{u})$

$a, b \in \mathbb{R}$

if $\lambda_{1,2} \in \mathbb{C}$: let $z(t) = e^{\lambda_1 t}\vec{v_1} \in \mathbb{C}^2$: complex sol.

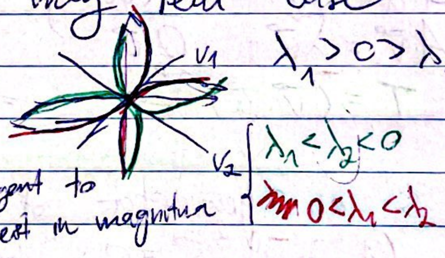
$e^{\lambda t} = e^{\alpha t}[\cos(\beta t) + i\sin(\beta t)]$

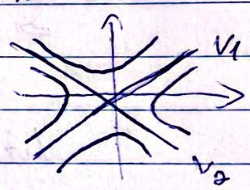
$\Rightarrow$ general real solution is $X(t) = a\,\mathrm{Re}(z) + b\,\mathrm{Im}(z)$

$a, b \in \mathbb{R}$

- When drawing phase portrait:
  ① mark the directions of eigenvectors

  ↳ in exam: only real case





Classification of planar systems

Trace - determinant plane

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a + d$ = Trace of A

- characteristic equation:

$$\det(A - \lambda I) = 0 \Leftrightarrow \lambda^2 - \underbrace{(a+d)}_{T = \mathrm{tr} A}\lambda + \underbrace{ad - bc}_{D := \det(A)} = 0$$

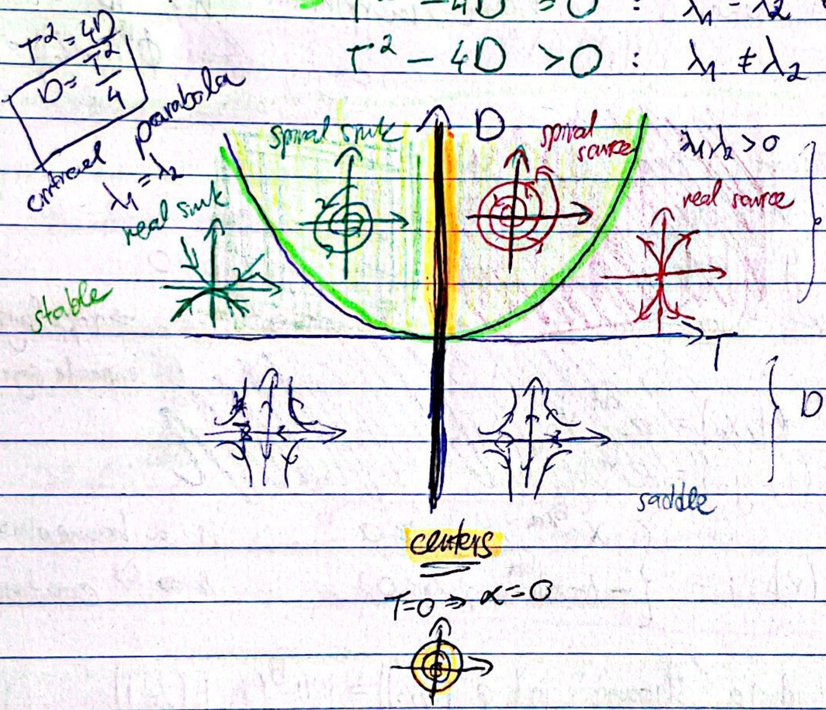
$$\Rightarrow \boxed{\lambda_{1,2} = \frac{T \pm \sqrt{T^2 - 4D}}{2}}$$

hence:

$T^2 - 4D < 0$: $\lambda_1 = \lambda_2^* \in \mathbb{C} \setminus \mathbb{R}$ $\quad \to D > \frac{T^2}{4}$

$T^2 - 4D = 0$: $\lambda_1 = \lambda_2 \in \mathbb{R}$

$T^2 - 4D > 0$: $\lambda_1 \neq \lambda_2$, $\lambda_{1,2} \in \mathbb{R}$



$\lambda_1 \lambda_2 > 0$ } unstable

$D > 0 \Rightarrow \mathrm{sign}(\lambda_1) = \mathrm{sign}(\lambda_2)$

$D < 0 \Rightarrow \mathrm{sign}(\lambda_1) \neq \mathrm{sign}(\lambda_2)$
$\Rightarrow$ saddle

Dynamical classification

- want categories : ~~such that~~ stable/unstable/saddle
- transformation - now not necessary linear anymore but still invertible and continuous

Def: Suppose $\vec{X}' = A\vec{X}$, $\vec{Y}' = B\vec{Y}$ have flows ϕ^A and ϕ^B.

Recall: Flow $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$

$(t, \vec{X}_0) \to \vec{X}(t)$ = solution such that $\vec{X}(0) = \vec{X}_0$

$\Rightarrow \phi(t, \vec{X}_0) = \vec{X}(t)$

$\phi^A(t, \vec{X}_0) = \exp(tA)\,\vec{X}_0 = x(t)$

The systems are called topologically conjugate if there is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$

conjugacy $\phi^B(t, h(\vec{X}_0)) = h(\phi^A(t, X_0))$

☆ consider two 1D systems

$x' = ax$, $x, y \in \mathbb{R}$

$y' = by$

Suppose a and b have the same sign ($a \cdot b > 0$). $\Rightarrow$ topological conj.

(if opposite signs $\Rightarrow$ not t.c.)

$\phi^a(t, x_0) = e^{at} x_0$

$\phi^b(t, y_0) = e^{bt} y_0$

! $h(x) := \begin{cases} x^{b/a}, & x \geq 0 \\ -(-x)^{b/a}, & x < 0 \end{cases}$ is a homeomorphism $\hookrightarrow$ continuous ✓

Need to show: $h(\phi^a(t, x_0)) \stackrel{!}{=} \phi^b(t, h(x_0))$

$h(\phi^a(t, x_0)) = h(e^{at} x_0) = \begin{cases} (e^{at} x_0)^{b/a}, & x_0 \geq 0 \\ -(e^{at} |x_0|)^{b/a}, & x_0 < 0 \end{cases}$

$= \begin{cases} e^{bt} x_0^{b/a}, & x_0 \geq 0 \\ -e^{bt} (-x_0)^{b/a}, & x_0 < 0 \end{cases} = e^{bt} \cdot \begin{cases} x_0^{b/a}, & x_0 \geq 0 \\ -(-x_0)^{b/a}, & x_0 < 0 \end{cases} =$

$= e^{bt}\, h(x_0) = \phi^b(t, h(x_0))$ □

& opposite signs

- $a > 0$, $b < 0$
  - if we ~~go~~ $t \to -\infty$ $\Rightarrow \phi^a \to 0 \Rightarrow h(\phi^a) = h(0)$

but ~~for~~ $e^{bt} \xrightarrow[t \to -\infty]{} \infty$

Def: A matrix A for the corresponding system $X' = AX$ is called **hyperbolic** if it has no eigenvalues on the imaginary axis (in particular $\neq 0$) $Re(\lambda) \neq 0$

↳ when perturbed slightly ⇒ remains hyperbolic

Thm: Let A, B be hyperbolic 2×2 matrices. Then $X' = AX$ and $Y' = BY$ are conjugate ⟺ A and B have the same number of eigenvalues with negative real part.

Remark: topological conjugacies define an equivalence relation

$$h\,\phi^A(t, X) = \phi^B(t, h(X))$$

h: homeomorphism on $\mathbb{R}^2$

- reflexivity $\phi^A \sim \phi^A$
- symmetry $\phi^A \sim \phi^B \Rightarrow \phi^B \sim \phi^A$
- transitivity $\phi^A \sim \phi^B \wedge \phi^B \sim \phi^C \Rightarrow \phi^A \sim \phi^C$

⇒ there are 3 categories of planar ~~linear~~ hyperbolic systems

Let $\lambda_{1,2}$ be eigenvalues of A:

1. $Re(\lambda_1) < 0$, $Re(\lambda_2) > 0$ (only possible for $\lambda_{1,2} \in \mathbb{R}$)
2. $Re(\lambda_1)$, $Re(\lambda_2) < 0$
3. $Re(\lambda_1)$, $Re(\lambda_2) > 0$

⇒ always topologically conjugate to its canonical form
(linear coordinate transformations which get us there are continuous and invertible → satisfy requirements of homeomorphism)

Lemma: $X' = AX$ is topologically conjugate to $Y' = \tilde{A}Y$ where $\tilde{A}$ is the canonical form of A.

Proof of Lemma: $X' = AX$ has flow $\phi^A(t, X_0) = X_0 e^{tA}$
$Y' = \tilde{A}Y$ has flow $\phi^{\tilde{A}}(t, Y_0) = Y_0 e^{t\tilde{A}}$

$\tilde{A}$ is canonical form of A ⇒ $\tilde{A} = T^{-1}AT$ where T is invertible 2×2 matrix.

[To be shown $\exists h: \mathbb{R}^2 \to \mathbb{R}^2$ homeomorphism s.t.
$\phi^A(t, h(X_0)) = h\,\phi^{\tilde{A}}(t, X_0)$]

$$\boxed{T^{-1}\exp(A)T = \exp(T^{-1}AT)} \quad \forall A, T$$

Let $h(x_0) = Tx_0$.

then, $\phi^A(t, h(x_0)) = \exp(tA)\, hx_0 = \exp(tA)\, Tx_0 =$

$$= T \underbrace{T^{-1}\exp(tA)\,T}_{=\exp(t\, T^{-1}AT)} x_0 = T\exp(t\tilde{A})x_0 =$$

$$= h(\exp(t\tilde{A})x_0) = h(\phi^{\tilde{A}}(t, x_0)) \quad \square$$

Proof of Thm:

Case 1: Suppose A has eigenvalues $\lambda_1 < 0 < \mu_1$

B has eigenvalues $\lambda_2 < 0 < \mu_2$.

have canonical forms:

$$\tilde{A} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \mu_1 \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \mu_2 \end{pmatrix}$$

$$\Rightarrow \begin{cases} x_1' = \lambda_1 x_1 \\ x_2' = \mu_1 x_2 \end{cases} \xrightarrow[\text{topologically conjugate}]{} \begin{cases} y_1' = \lambda_2 y_1 \\ y_2' = \mu_2 y_2 \end{cases}$$

as shown for 1D case on p. 18

Case 2: B, A have canonical forms:

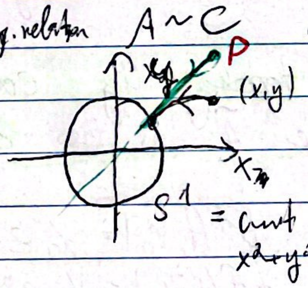
$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \quad \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

with $\alpha < 0$ $\qquad$ $\lambda, \mu < 0$ $\qquad$ $\lambda < 0$

② [Show conjugacy to $C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, then

eq. relation $A \sim C$ and $C \sim B$ $\Rightarrow$ $A \sim B$.

idea: $X = \binom{x}{y}$, $X' = AX$



(x,y) - start here, then propagate the solution using the flow ϕ^A → ~~this will~~ until it reaches the unit circle. The time needed for this $= \tau$

S^1 = unit circle $x^2 + y^2 = 1$

$$\phi^A(t, X_0) = \phi^A(t, (x,y)) = \phi^A(\tau, (x,y)) =: \phi^A_\tau((x,y))$$

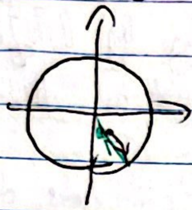
↳ exists because all solutions tend towards $(0,0)$ as $t \to \infty$ because Re(eigenvalues) < 0

Flow lines of C are straight lines through the origin

$$P = \phi^C_{-\tau} \circ \phi^A_\tau(x,y) = h(x,y)$$

What if we start inside the unit circle? $\tau < 0$

$\Rightarrow$ propagate back in time



definition $\Rightarrow -\tau > 0$

$\Rightarrow$ also works for points within the unit circle (just $\tau < 0$)

↳ only point where it doesn't work is $(0,0)$

$\Rightarrow$ equilibrium point, never going to go anywhere

↳ define separately

$$\boxed{h(x,y) = \begin{cases} \phi^C_{-\tau} \circ \phi^A_{\tau}(x,y) & , (x,y) \neq (0,0) \\ (0,0) & , (x,y) = (0,0) \end{cases}}$$

First, we show that $h(\phi^A_s(x,y)) = \phi^C_s(h(x,y))$ $\forall (x,y) \in \mathbb{R}^2$, $\forall s \in \mathbb{R}$ (time)

- $\tau(\phi^A_s(x,y)) = \tau(x,y) - s$

  ↳ time needed for point $\phi^A_s(x,y)$ to reach the unit circle

- $(x,y) \neq (0,0)$:

$\phi^A_{\tau_1+\tau_2} = \phi^A_{\tau_2} \circ \phi^A_{\tau_1}$

$$h(\phi^A_s(x,y)) = \phi^C_{-\tau+s} \circ \phi^A_{\tau(x,y)-s} \circ \phi^A_s(x,y) = \phi^C_s \circ \underbrace{\phi^C_{-\tau(x,y)} \circ \phi^A_{\tau(x,y)}(x,y)}_{h(x,y)} = \phi^C_s\, h(x,y)$$

- $(x,y) = (0,0)$ holds trivially ✓

$$h(\phi^A_s(0,0)) = (0,0) = \phi^C_s(h(0,0)) = \phi^C_s((0,0))$$

Next, we need to show that h is invertible continuous map.

- invertible:

$$\boxed{g(x,y) = h^{-1}(x,y) = \begin{cases} \phi^A_{-\tau'} \circ \phi^C_{\tau'}(x,y) & , (x,y) \neq (0,0) \\ (0,0) & , (x,y) = (0,0) \end{cases}}$$

τ' is the time the solution starting at (x,y) takes to reach the unit circle S^1

↳ $\tau'(x,y) = \ln(r)$, $r = \sqrt{x^2+y^2} = r_0 e^{-t} \equiv 1$

solutions of system $\begin{pmatrix} x \\ y \end{pmatrix}' = C \begin{pmatrix} x \\ y \end{pmatrix}$

↳ eigenvalues -1

↳ $x = x_0 e^{-t}$, $y = y_0 e^{-t}$

- continuity of τ':

  – at $(x,y) \neq (0,0)$ $\Rightarrow$ composition of continuous maps

Let $(x_n, y_n) \neq (0,0)$ sequence s.t. $(x_n, y_n) \to (0,0)$ as $n \to \infty$

To show: $g(x_n, y_n) \to g(0,0) = (0,0)$ as $n \to \infty$

$\Rightarrow \tau'(x_n, y_n) \xrightarrow[n\to\infty]{} -\infty$

let $(\tilde{x}_n, \tilde{y}_n) = \phi_{\tau'}(x_n, y_n) \in S^1 \Rightarrow \phi^A_{-\tau'(x_n,y_n)}(\tilde{x}_n, \tilde{y}_n) \xrightarrow[n\to\infty]{} (0,0)$

$\frac{\partial F}{\partial t}$

- continuity of h:

$\sigma = \sigma(x,y)$ is defined implicitly by $\|\phi_\sigma^A(x,y)\|=1$

$\rightsquigarrow$ solve? use implicit function theorem

can be solved for t $\Leftrightarrow$ $\frac{\partial}{\partial t}\|\phi_t^A(x,y)\| \neq 0$ by

↳ also implies continuous differentiability

$$(x(t),y(t)) = \phi_t^A(x,y) = \frac{\partial}{\partial t}\sqrt{x^2(t)+y^2(t)} =$$

$$= \frac{x(t)x'(t)+y(t)y'(t)}{\sqrt{x^2(t)+y^2(t)}} = \begin{pmatrix} \frac{x(t)}{r(t)} \\ \frac{y(t)}{r(t)} \end{pmatrix} \Big\| \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

point on the unit circle S^1 — vector field at the point on S^1: $\begin{pmatrix} x' \\ y' \end{pmatrix} = A\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$$\frac{\partial}{\partial t}\phi_t^A \neq 0 \iff \begin{pmatrix} x' \\ y' \end{pmatrix} \not\perp \begin{pmatrix} x \\ y \end{pmatrix}$$

↳ vector field is not tangent to the unit circle anywhere on the circle.

$n_1^2+n_2^2=1$, $\begin{pmatrix} n_1 \\ n_2 \end{pmatrix}$

$$\frac{\partial}{\partial t}\phi_t^A = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \cdot A\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \overset{a}{=} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}\begin{pmatrix} \lambda n_1 \\ \mu n_2 \end{pmatrix} = \lambda n_1^2 + \mu n_2^2 < 0 \checkmark$$

(a) case: $A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, $\lambda,\mu<0$

(b) case $A = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $\alpha<0$

$$\frac{\partial}{\partial t}\phi_t^A \overset{b}{=} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}\begin{pmatrix} \alpha n_1+\beta n_2 \\ \alpha n_2 - \beta n_1 \end{pmatrix} = \alpha n_1^2 + \beta n_1 n_2 + \alpha n_2^2 - \beta n_1 n_2 = \alpha(n_1^2+n_2^2) < 0 \checkmark$$

(c) case $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda<0$

$$\frac{\partial}{\partial t}\phi_t^A \overset{c}{=} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}\begin{pmatrix} \lambda n_1+n_2 \\ \lambda n_2 \end{pmatrix} = \lambda n_1^2 + n_1 n_2 + \lambda n_2^2 \overset{?}{\neq} 0$$

$$\iff \lambda\underbrace{(n_1^2+n_2^2)}_{=1} \neq -n_1 n_2$$

$$\iff \lambda \neq -n_1 n_2$$

↳ not true in general !!

$T = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}$, $\varepsilon>0$ but $\varepsilon \ll 1$.

$T^{-1}AT = \begin{pmatrix} \lambda & \varepsilon \\ 0 & \lambda \end{pmatrix}$ $\rightsquigarrow$ valid thing to do !!

$$\Rightarrow \frac{\partial}{\partial t}\phi_t^A = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}\begin{pmatrix} \lambda n_1+\varepsilon n_2 \\ \lambda n_2 \end{pmatrix} = \lambda n_1^2 + \varepsilon n_1 n_2 + \lambda n_2^2 = \lambda + \varepsilon n_1 n_2$$

if ε sufficiently small

$\Rightarrow -\varepsilon n_1 n_2 < \lambda \Rightarrow < 0$ ✓

Higher dimensional linear systems

Canonical forms

Prop: Let A be $n \times n$ matrix. Then $\exists$ non-singular (invertible) matrix T such that

$$T^{-1}AT = \begin{pmatrix} B_1 & & 0 \\ & B_2 & \\ 0 & & \ddots B_k \end{pmatrix}, \quad k \leq n,$$

↳ block matrices

where B_j are square matrices of the following form:

$\lambda \in \mathbb{R}$ i) $\begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$ for $\lambda \in \mathbb{R}$

size depends on dimensions of eigenspace

$\lambda \in \mathbb{C}$ or ii) $\begin{pmatrix} C_2 & I_2 & & 0 \\ & C_2 & I_2 & \\ & & \ddots & I_2 \\ 0 & & & C_2 \end{pmatrix}$ ← for $\lambda \in \mathbb{C}$

where $C_2 = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\alpha, \beta \in \mathbb{R}$

$\Rightarrow$ real Jordan-normal form (can be just i) for $\lambda \in \mathbb{C}$ but we want real matrix $\Rightarrow$ for $\lambda \in \mathbb{C}$ use ii))

$\Rightarrow$ B_j can also be $= (\lambda)$ or $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

1x1 matrix

Genericity

matrices with distinct eigenvalues are very common in $L(\mathbb{R}^n)$, set of all $n \times n$ matrices

Thm: The set of matrices in $L(\mathbb{R}^n)$ that have distinct eigenvalues is open and dense in $L(\mathbb{R}^n)$

Proof: We need a topology in $L(\mathbb{R}^n)$. To this end identify $L(\mathbb{R}^n)$ with the vector space $\mathbb{R}^{n^2}$ and define a norm on $L(\mathbb{R}^n)$ from some norm on $\mathbb{R}^{n^2}$ (which one all equivalent).

$A \in L(\mathbb{R}^n)$. Chose norm: $\|A\|_\infty := \max_{ij} |a_{ij}|$

↳ largest entry in absolute terms

Let the set of $n \times n$ matrices with distinct eigenvalues be denoted by M.

<u>M is dense in $L(\mathbb{R}^n)$</u>

Let $A \in L(\mathbb{R}^n)$ and $\varepsilon > 0$.

<u>Show:</u> $\exists B \in M$ s.t. $\|A-B\|_\infty < \varepsilon \Rightarrow$ ~~A and B are~~

↳ there is another matrix from M in any small neighborhood around A.

<u>Pf:</u> Consider the canonical form of A, i.e. $\exists$ non-singular T s.t. $T^{-1}AT$ has canonical form.

For canonical form "slightly" change diagonal elements to get a matrix with distinct eigenvalues.

use that $\varphi: L(\mathbb{R}^n) \to L(\mathbb{R}^n)$, $A \mapsto \underbrace{T^{-1}AT}_{\text{canonical form}}$ is continuous and invertible.

↳ $A \overset{\varphi}{\mapsto} T^{-1}AT \approx \tilde{B} \xrightarrow{\varphi^{-1}} B$, then from continuity of φ, B is close to A

i.e. $\|A-B\|_\infty < \varepsilon$, $\forall \varepsilon > 0$.

<u>M is open in $L(\mathbb{R}^n)$</u>, let $A \in M$.

<u>Show:</u> $\varepsilon > 0$, s.t. $\|A-B\| < \varepsilon$ implies $B \in M$

$L(\mathbb{R}^n) \to$ polynomials of degree n (finite dim. VS)

$A \mapsto p(\lambda) = \det(A - \lambda I) = a_0 + a_1\lambda + \dots + a_n\lambda^n$

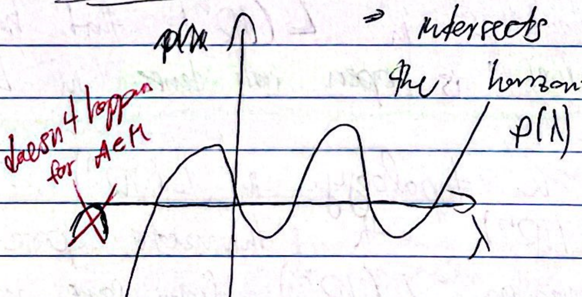
continuous

↳ characteristic polynomial

Define norm $\|p\| = \sqrt{a_0^2 + a_1^2 + \dots + a_n^2}$

<u>If $A \in M$:</u> $\Rightarrow A$ has distinct eigenvalues

$\Rightarrow$ intersects transversely (never just "touches") the horizontal axis



If $\|p - q\| < \varepsilon$ for some $\varepsilon > 0$, q has distinct zeroes.

$\Rightarrow$ By continuity $\exists \delta$ s.t. $\|B - A\| < \delta$, then $B \in M$. □

(B: matrix corresponding to q)

$\Rightarrow$ set M is "very large" in $L(\mathbb{R}^n)$

Non-linear systems

Existence and Uniqueness of solutions

Existence Thm (Peano): • $D \subset \mathbb{R}^{n+1}$ domain $= \mathbb{R} \times \mathbb{R}^n$ (time, space)

• $F: D \to \mathbb{R}^n$ continuous
$(t, x) \mapsto F(t,x)$

• $X' = F(t,X)$ with initial condition $X(t_0) = X_0$ where $(t_0, X_0) \in D$

F continuous $\Rightarrow X' = F(t,X)$ has a solution, which can be extended to the boundary of D

Continuity of $F \Rightarrow$ existence
$\not\Rightarrow$ uniqueness

↳ example: $x' = 3x^{2/3}$, $x \in \mathbb{R}$ ⇝ continuous ✓
$x(t) \equiv 0$ is a solution with $x(0)=0$
$x(t) = t^3$ is also a solution with $x(0)=0$ } not unique

Picard-Lindelöf Thm: • $D \subset \mathbb{R}^{n+1}$ domain

• $F: D \to \mathbb{R}^n$ continuous and satisfying **locally** Lipschitz condition w.r.t. X. i.e. $\forall (t_0, X_0) \in D$, $\exists K \geq 0$ and open neighbourhood of (t_0, X_0) s.t.
$\|F(t, X) - F(t, Y)\| \leq K \cdot \|X - Y\|$ $\forall X, Y$ in the open neighborhood

↳ there's a bound (estimate) on the distance between $F(X)$ and $F(Y)$ wrt the distance between X and Y

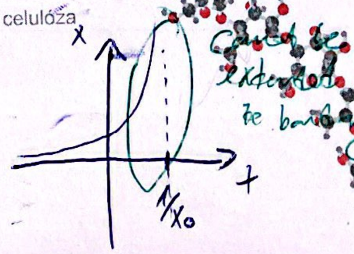
stronger than continuity but not necessarily entailing differentiability

Thm $X' = F(t,X)$, $X(t_0) = X_0$ with $(t_0, X_0) \in D$ has a unique solution (that can be extended to the boundary of D)

Ex $x' = x^2$, $x(0) = x_0$ — solution not def...

$\int \frac{dx}{x^2} = \int dt$

$-\frac{1}{x} - \left(-\frac{1}{x_0}\right) = t \Rightarrow -\frac{1}{x} + \frac{1}{x_0} = t \Rightarrow \boxed{x(t) = \ldots}$



Continuous dependence on initial conditions & parameters

Thm: Let $O \subset \mathbb{R}^n$ open and let $F: O \to \mathbb{R}^n$ be Lipschitz with Lipschitz constant $K \geq 0$ and $y(t), z(t)$ be solutions of $X' = F(X)$ with $y(t), z(t) \in O \quad \forall t \in [t_0, t_1]$.
Then $\|y(t) - z(t)\| \leq \|y(t_0) - z(t_0)\| e^{K(t-t_0)} \quad \forall t \in [t_0, t_1]$

difference cannot grow faster than exponentially fast

doesn't depend explicitly on t

↳ Proof on p. 26.

Corollary: Let ϕ flow of $X' = F(X)$ with F as above.
Then ϕ_t continuous $\forall t \in [t_0, t_1]$

Thm: Grönwall inequality
$u: [0, \alpha] \to \mathbb{R}$ continuous and non-negative.
↳ $u(t) \geq 0 \quad \forall t \in [0, \alpha]$

Suppose $C, K \geq 0$ s.t. $u(t) \leq C + \int_0^t K u(s)\,ds$
Then $\Rightarrow u(t) \leq Ce^{Kt} \quad \forall t \in [0, \alpha]$

Proof: Suppose $C > 0$. Set $\tilde{u}(t) := \underbrace{C}_{>0} + \underbrace{\int_0^t K u(s)\,ds}_{\geq 0} > 0$
and so $u(t) \leq \tilde{u}(t)$ by assumption.

$\tilde{u}'(t) = K u(t)$

$\frac{\tilde{u}'(t)}{\tilde{u}(t)} = \frac{K u(t)}{\tilde{u}(t)} \leq K$ because $u(t) \leq \tilde{u}(t)$

$\Rightarrow \frac{d}{dt} \ln(\tilde{u}(t)) \leq K$ integrate to get $\ln(\tilde{u}(t)) - \ln(\tilde{u}(0)) \leq Kt - K \cdot 0$

integrate both sides from 0 to t

$\ln(\tilde{u}(t)) \leq Kt + \ln(\tilde{u}(0)) = Kt + \ln(C)$ (from above)

$\Rightarrow \tilde{u}(t) \leq Ce^{Kt}$

$u(t) \leq \tilde{u}(t) \leq Ce^{Kt} \Rightarrow \boxed{u(t) \leq Ce^{Kt}}$

For $C = 0$: consider convergent sequence of C_n's with $C_n \searrow 0$ as $n \to \infty$. ⌑

Remark: [if $u(t) \leq \int_0^t K u(s)\,ds \Rightarrow u(t) \leq 0$ but $u(t) \geq 0 \Rightarrow u(t) = 0$
↳ very strong condition]

Proof of Thm from top of p.25:

(from top of p.25)

• set $v(t) := \| y(t) - z(t) \|$.

We have $\|y(t) - z(t)\| = \| y(t_0) - z(t_0) + \int_{t_0}^{t} F(y(s)) - F(z(s))\, ds \|$

$\Rightarrow$ triangular inequality

$\|y(t) - z(t)\| \leq \|y(t_0) - z(t_0)\| + \int_{t_0}^{t} \|F(y(s)) - F(z(s))\|\, ds$

$\Rightarrow \quad v(t) \leq v(t_0) + \int_{t_0}^{t} K v(s)\, ds$

$\leq K\|y(s) - z(s)\|$ by Lipschitz condition

• set $u(t) := v(t + t_0)$

↳ to shift our interval from $[t_0, t_1]$ to $[0, \alpha]$ as required by Grönwall

• By Grönwall's inequality:

pre-requisite for Grönwall's ineq. $\left\{ u(t) = v(t+t_0) \leq v(t_0) + \int_{t_0}^{t_0+t} K\, v(s)\, ds \underset{\sigma = s - t_0}{=} v(t_0) + \int_0^t K\, v(\sigma)\, d\sigma \right.$

~~insert~~

$\Rightarrow$ now applying: $u(t) = v(t+t_0) \leq v(t_0)\, e^{Kt}$

$\Rightarrow \quad v(t) \leq v(t_0)\, e^{K(t-t_0)}$ ◻

⇝ also applies to chaotic systems (if Lipschitz holds)

↳ but still an exponential growth possible

⇝ but we have that if $\|y(t) - z(t)\|\, e^{K(t-t_0)} \to 0$

then also $\|y(t) - z(t)\| \to 0$.

Thm: continuous dependence on parameters

$X' = F_a(\bar{X})$, $\bar{X} \in \mathbb{R}^n$, $a \in \mathbb{R}^m$ (parameters)

If F is Lipschitz wrt both $\bar{X}$ and a

$\Rightarrow \phi_t^a$ (flow of F_a) depends continuously on both $\bar{X}$ and a.

Proof: consider augmented system

$$(*) \left\{ \begin{array}{l} X' = F_a(X) \\ a' = 0 \end{array} \right.$$

$\Rightarrow$ we view parameters as dynamical variables, constant in time

$F_a(X) = (f_1(\overbrace{x, a}^{\tilde{x}}), \ldots, f_n(x, a))$

Let $\tilde{X} = (x_1, x_2, \ldots, x_n, a_1, \ldots, a_m) \in \mathbb{R}^{n+m}$ extended state space

$\Rightarrow \tilde{F}(\tilde{X}) = (f_1(\tilde{X}), \ldots, f_n(\tilde{X}), \underbrace{f_{n+1}(\tilde{X}), \ldots, f_{n+m}(\tilde{X})}_{=0})$

(*) $\Leftrightarrow$ $\tilde{\chi}' = \tilde{F}(\tilde{x})$

$F(\tilde{x})$ satisfies Lipschitz condition.

Now apply previous result $\Rightarrow$ continuous dep. of ϕ on $\tilde{x}$ $\square$

- $F \in C^1 \Rightarrow F$ satisfies Lipschitz condition

Lemma: $O \subset \mathbb{R}^n$ open
$F: O \to \mathbb{R}^n$ is of class C^1.
$\Rightarrow$ Then F is locally Lipschitz.

Proof: Let $\bar{x}_0 \in O$. To be shown:

[$\exists$ open neighbourhood $= O(x_0)$ and $K \geq 0$
s.t. $\| F(\bar{y}) - F(\bar{x}) \| \leq K \| \bar{x} - \bar{y} \|$ $\forall \bar{x}, \bar{y} \in O(x_0)$]

Choose $\varepsilon > 0$ s.t. $\underbrace{B_\varepsilon(x_0)}_{\text{closed ball of radius } \varepsilon \text{ centered at } x_0} \subseteq O$ $\rightsquigarrow |\bar{x} - \bar{x}_0| \leq \varepsilon$

Set $K := \max \underbrace{\| DF(x) \|}_{\substack{\text{Jacobian of } F(x) \\ (n \times n)}}$, $\bar{x} \in B_\varepsilon(\bar{x}_0)$ (By Weierstraß Thm: this max K is attained on the closed ball.)

$\Rightarrow K < \infty$ as DF is continuous and $B_\varepsilon(x)$ is compact.
follows by Weierstrass Thm

Let $X, Y \in B_\varepsilon(\bar{x}_0)$,

$B_\varepsilon(\bar{x}_0)$ is convex $\Rightarrow$ $[(1-s)\bar{x} + s\bar{y}] \in B_\varepsilon(\bar{x}_0)$, $\forall s \in [0,1]$
$\phantom{B_\varepsilon(\bar{x}_0) \text{ is convex } \Rightarrow} \overset{\shortparallel}{\bar{x} + s(\bar{y} - \bar{x})}$

Set $\psi(s) := F(\bar{x} + s(\bar{y} - \bar{x}))$

and $\underbrace{O(x_0)}_{\text{open neighbourhood}} := \underbrace{B_\varepsilon(x_0)}_{}$ open ball = interior of $B_\varepsilon(x_0)$, i.e. $\forall x$ s.t. $\|x - x_0\| < \varepsilon$

$\|F(x) - F(\bar{y})\| = \| \psi(1) - \psi(0) \| \overset{\text{mean value Thm}}{=} \| \psi'(s)(1-0) \|$ for some $s \in [0,1]$

$\psi'(s) = \underbrace{DF(\bar{x} + s(\bar{y} - \bar{x}))}_{F'} \cdot \underbrace{(\bar{y} - \bar{x})}_{\text{inner derivative}}$ $\leftarrow$ chain rule

$\|F(\bar{x}) - F(\bar{y})\| = \| DF(\bar{x} + s(\bar{y} - \bar{x})) \cdot (\bar{y} - \bar{x}) \cdot 1 \| \leq K \| \bar{y} - \bar{x} \|$

because $\|DF\| \leq K$ for some K

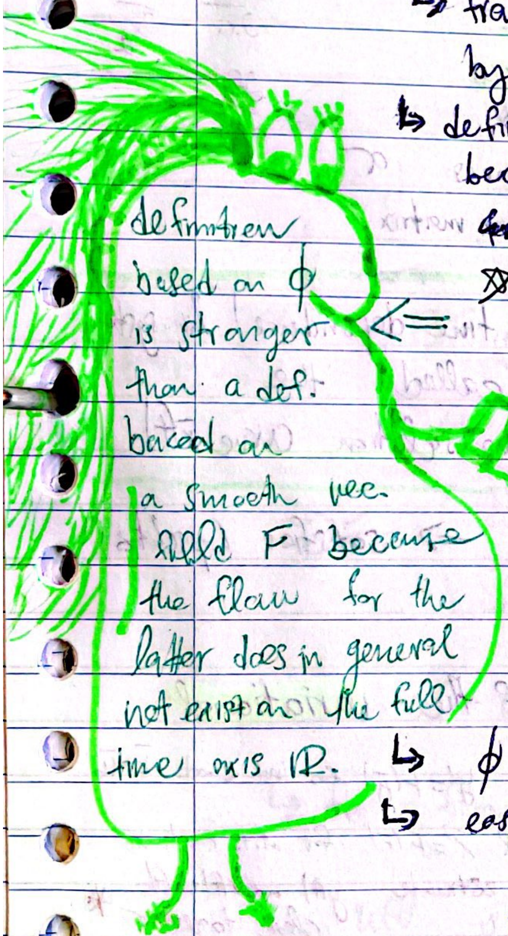
Thm: smoothness of flows
Consider $x' = F(x)$ where F is C^1.
Then the flow ϕ is also C^1, i.e.
$\left(\frac{\partial \phi}{\partial t} \text{ and } \frac{\partial \phi}{\partial x} \text{ exist and are continuous}\right)$

$\frac{\partial \phi}{\partial t} = F(x)$ □

Def: A smooth dynamical system on $\mathbb{R}^n$ is a continuously differentiable function $\phi: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $(t, x) \mapsto \phi(t, x) = \phi_t(x) = \phi^t(x)$
satisfying:
1. $\phi_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the identity.
2. $\phi_s \circ \phi_t = \phi_{s+t}$ $\forall s, t \in \mathbb{R}$

can be extended to discrete if we take $\phi: \mathbb{Z} \times \mathbb{R}^n \to \mathbb{R}^n$.

↳ from here we can define a vector field by taking a time derivative of ϕ: $\frac{\partial \phi}{\partial t} = F$
↳ definition in terms of ϕ rather than F, because F is ~~often not~~ generally not defined $\forall t \in \mathbb{R}$



definition based on ϕ is stronger than a def. based on a smooth vec. field F because the flow for the latter does in general not exist on the full time axis $\mathbb{R}$. $\Leftarrow$

Ex: $F: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, F is C^1 ✓
but $x' = F(x) = x^2$ has solution
$\boxed{x(t) = \frac{1}{\frac{1}{x_0} - t}, \; x_0 = x(0)}$
$\boxed{x(t) = \frac{x_0}{1 - t x_0}}$

when $t = \frac{1}{x_0}$, solution diverges (has singularity)
$\Rightarrow$ solution $x(t)$ doesn't exist $\forall t \in \mathbb{R}$.

↳ ϕ defines an $\mathbb{R}$-action ⇝ homomorphism wrt t.
↳ easy to find inverse:

$$\phi_t \circ \phi_t^{-1} = id = \phi_0$$
$$\Rightarrow \phi_t \circ \phi_t^{-1} = \phi_0 \quad \Rightarrow \quad \phi_t \circ \phi_s = \phi_{t+s} =: \phi_0$$
$$\Rightarrow t + s = 0 \quad \Rightarrow s = -t$$
$$\Rightarrow \phi_t^{-1} = \phi_{-t} \quad \rightsquigarrow \quad (\phi_t)^{-1} = \phi_{-t}$$

↳ ϕ_t is C^1 for any $t \in \mathbb{R}$ and — " — $\forall t \in \mathbb{R}$
$\Rightarrow$ ϕ_t is a diffeomorphism = bijective homeomorphism, cont. w/ cont. inverse

we have: $\boxed{\frac{\partial\phi}{\partial t}(t,\bar{x}) = F(\phi(t,x))}$ when ϕ corresponds to flow of vector field

What about $\frac{\partial\phi}{\partial x}$?

$\boxed{\frac{\partial\phi}{\partial x} = D\phi_t}$ where $D\phi_t$ is the Jacobian matrix of $\bar{x} \mapsto \phi_t(\bar{x})$ for fixed t.

(?) dynamical and geometric meaning of $D\phi_t$

⇒ **Variational equations**

- Consider $\bar{x}' = F(\bar{x})$ with solution $\bar{x}(t)$ on time interval $J = [a,b]$ ⇒ compact interval with $\bar{x}(t_0) = \bar{x}_0$, $t_0 \in J$ and F a C^1 vector field on $\mathbb{R}^n$.

$$A(t) := DF(\bar{x}(t)) = DF_{\bar{x}(t)}$$

$\frac{\partial f_1}{\partial x_1}$ $\frac{\partial f_2}{\partial x_2}$ $\frac{\partial f_2}{\partial x_2}$

↳ $n \times n$ Jacobian matrix

For $t \in [a,b]$, cont. family of matrices or a function mapping $t \mapsto n\times n$ matrix

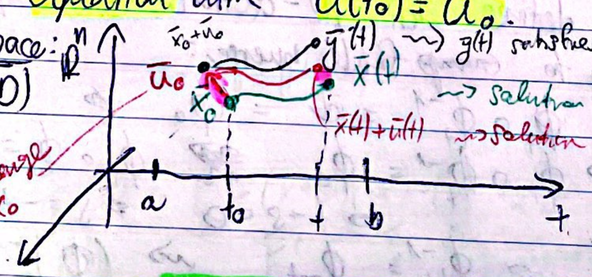
always for particular solution

Def: The non-autonomous (explicit time dependence) lin. system $u' = A(t)u$, $u \in \mathbb{R}^n$ is called the variational equation along the solution curve $\bar{x}(t)$.

↳ A depends continuously on t ⇒ satisfies Lipschitz condition ⇒ unique solutions

- Let $u(t)$ be the solution of the variational equation with $u(t_0) = u_0$.

State space: $\mathbb{R}^n$ (now 2D)



small change in $\bar{x}_0$

⇝ $y(t)$ satisfies $y' = F(\bar{y})$ for init. cond. $\bar{x}_0 + \bar{u}_0$

⇝ solution of $\bar{x}' = F(\bar{x})$ for init. cond. x_0

⇝ solution estimate: $y(t)$ and $x(t) + u(t)$ close together *

*and converge very fast

in fact: $\boxed{\lim_{u_0 \to 0} \frac{1}{\|u_0\|} \|y(t) - (x(t) + u(t))\| = 0}$ ↙ indep. of t holds uniformly $\forall t \in J$.

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$\underline{\underline{\text{Th}^{m}}}$: Under the conditions above,

$\underbrace{D\phi_t(x_0)}_{\text{matrix}} \cdot u_0 = u(t, u_0)$ where $u(t,u_0)$ is the solution of the variational equation

vectors ✓

$u' = \underbrace{D F_{x(t)}}_{=: A(t) \text{ for given } x(t)} u$ with $u(t_0) = u_0$

Proof: $D\phi_t \cdot u_0 = \lim_{h \to 0} \frac{\phi_t(x_0 + h u_0) - \phi_t(x_0)}{h} = *$

'directional derivative'

$= \lim_{h \to 0} \frac{u(t, h u_0)}{h} = \lim_{h \to 0} \frac{\cancel{h} u(t, u_0)}{\cancel{h}} = u(t, u_0)$ ▫

$* \lim_{u_0 \to 0} \frac{1}{\|u_0\|} \| y(t) - (x(t) + u(t)) \| = 0 \Rightarrow \lim_{u_0 \to 0} \frac{1}{\|u_0\|} \| (y(t) - x(t)) - u(t) \| = 0$

$D\phi_t(x_0)$ maps u_0 to the deviation after time t.

deviation of $x(t)$ and $y(t)$ at time t_0

☆ $x' = x + y^2$, $\bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$y' = -y$

Take solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\forall t$

↳ equilibrium

~~Choose~~

$F = \begin{pmatrix} x + y^2 \\ -y \end{pmatrix}$ $\underbrace{DF_{\binom{x(t)}{y(t)}}}_{A(t)} = DF_{\binom{0}{0}} = \begin{pmatrix} 1 & 2y \\ 0 & -1 \end{pmatrix} \Big|_{(0,0)} = \underline{\underline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}}$

$$\boxed{DF = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \vec{\nabla} f_1 \\ \vec{\nabla} f_2 \\ \vdots \end{pmatrix}}$$

$\Rightarrow u' = DF_{\binom{0}{0}} u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u \Rightarrow u(t) = \begin{pmatrix} x_0 e^{t} \\ y_0 e^{-t} \end{pmatrix}$ for $u_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

↳ for a fixed interval $[-\tau, \tau]$ ($0 \in [\tau, \tau]$ ✓)

$\tau > 0$ fixed

$u(t)$ is very close to $x(t)$ where $x' = F(x)$ with $\bar{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

Equilibria of nonlinear systems

$\bar{x}' = F(\bar{x})$, suppose $F \in C^{\infty}(\mathbb{R}^n)$, $\bar{x} \in \mathbb{R}^n$

$\bar{x}_0$ is an equilibrium

i.e. $F(\bar{x}_0) = 0$

$\bar{x}(t) = \bar{x}_0 + \underbrace{\Delta\bar{x}(t)}_{\bar{u}(t)}$, for $\Delta\bar{x}(t)$ "small"

$$\frac{d}{dt}\bar{x}(t) = F(\bar{x}_0 + \Delta\bar{x}(t)) = \underbrace{F(\bar{x}_0)}_{=0} + DF(\bar{x}_0)\Delta\bar{x}(t) + O(\Delta\bar{x})$$

$$\frac{d}{dt}(\bar{x}_0 + \Delta\bar{x}(t)) = \frac{d}{dt}\Delta\bar{x}(t) \quad \text{because } \bar{x}_0 \text{ is equilibrium} \Rightarrow \bar{x}_0' = 0 = F(\bar{x}_0)$$

$$\Rightarrow \boxed{\frac{d}{dt}\underbrace{\Delta\bar{x}(t)}_{u} \cong \underbrace{DF(\bar{x}_0)}_{A} \underbrace{\Delta\bar{x}(t)}_{u}}$$

neglect higher order terms $O(\Delta\bar{x})$

linearisation at an equilibrium point

- we want: linearisation to be good approximation for the nonlinear system around the equilibrium

? What does the linearised system tell us about the original nonlinear system?

Ex
$$\begin{cases} x' = x + y^2 \\ y' = -y \end{cases}, \qquad \bar{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only equilibrium.

$\frac{d}{dt}\bar{x} = F(\bar{x})$

$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y^2 \\ -y \end{pmatrix}$

↳ we need $x' = 0$ and $y' = 0$.

$\Rightarrow (y' = 0 \Leftrightarrow -y = 0 \Leftrightarrow y = 0)$

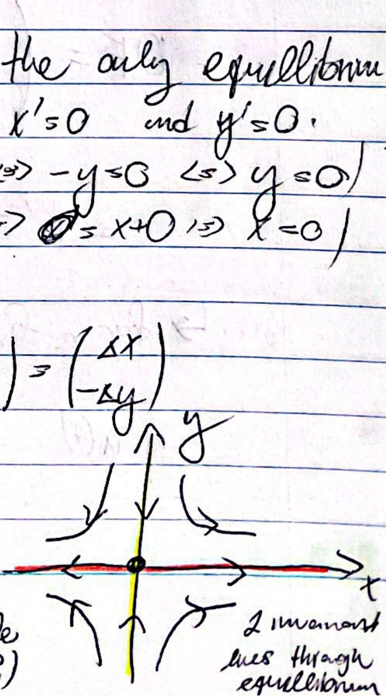
$\Rightarrow (x' = 0 \Leftrightarrow 0 = x + 0 \Rightarrow x = 0)$

linear:
$$\frac{d}{dt}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = DF_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$

$$\frac{d}{dt}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 1 & 2y \\ 0 & -1 \end{pmatrix}_{\begin{pmatrix} 0 \\ 0 \end{pmatrix}} \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} = \begin{pmatrix} \Delta x \\ -\Delta y \end{pmatrix}$$

$$\Rightarrow \begin{cases} \Delta x' = \Delta x \\ \Delta y' = -\Delta y \end{cases} \Rightarrow \boxed{\begin{aligned} \Delta x(t) &= \Delta x_0 e^{t} \\ \Delta y(t) &= \Delta y_0 e^{-t} \end{aligned}}$$

⇝ saddle at $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$



For non linear system:

$$e^{-\int f(t)\,dt} = e^{-\int 1\,dt} = e^{-t}$$

$$\begin{cases} x' = x + y^2 \\ y' = -y \end{cases} \Rightarrow \quad y(t) = y_0 e^{-t}$$

$$x' = x + y_0^2 e^{-2t}$$

(I) particular sol:

$$x(t) = Ke^{-2t}$$

$$-2Ke^{-2t} = Ke^{-2t} + y_0^2 e^{-2t}$$

$$-3K = y_0^2$$

$$K = -\frac{1}{3} y_0^2 \Rightarrow$$

$$\Rightarrow x_p(t) = -\frac{1}{3} y_0^2 e^{-2t}$$

homog. sol. $x(t) = Ce^{t}$

$$\Rightarrow \boxed{x(t) = Ce^{t} - \frac{1}{3} y_0^2 e^{-2t}}$$

$$\Rightarrow x' = x + y_0^2 e^{-2t}$$

$$x' - x = y_0^2 e^{-2t}$$

(II) $x'e^{-t} - e^{-t} = y_0^2 e^{-3t}$

$$(xe^{-t})' = y_0^2 e^{-3t}$$

$$xe^{-t} = \int y_0^2 e^{-3t}\,dt$$

$$xe^{-t} = y_0^2 \frac{e^{-3t}}{-3} + C$$

$$\boxed{x = -\frac{y_0^2}{3} e^{-2t} + Ce^{t}}$$

C determined by (x_0, y_0): $x(0) = C - \frac{1}{3} y_0^2$

$$\Rightarrow C = x_0 + \frac{1}{3} y_0^2$$

$$\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (x_0 + \frac{1}{3} y_0^2) e^{t} - \frac{1}{3} y_0^2 e^{-2t} \\ y_0 e^{-t} \end{pmatrix}$$

celulóza

*alternatively:

parametrise curve by

$$r: y \mapsto (-\tfrac{1}{3} y^2, y)$$

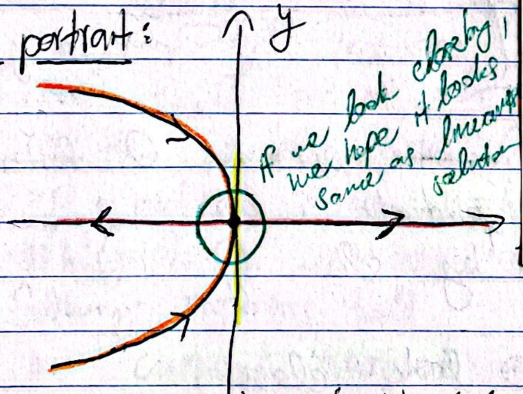
- ODE's $\Rightarrow$ vector field.
  - curve invariant in vector field $\Leftrightarrow$ vector field is tangent (time doesn't effect it)
- tangent vector: $r'(y) = (-\frac{2}{3} y, 1)$

Show: $F(x,y) \parallel r'(y)$

vector field $\parallel$ tangent vector

VŠCHT PRAHA

Phase portrait:



the invariant line with increasing sol.

- Here $x_0 = 0$ (y-axis) $\Rightarrow \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{3} y_0^2 (e^{t} - e^{-2t}) \\ y_0 e^{-t} \end{pmatrix}$

doesn't remain there $\Rightarrow$ y-axis is not invariant for the non linear system.

The value of this function doesn't change in time

- instead second invariant line is $\boxed{x + \frac{1}{3} y^2 = 0} = f(x(t), y(t))$

Proof:* Let $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ which at some point is on the curve, i.e. $x_0 + \frac{1}{3} y_0^2 = 0$

$\Rightarrow$ To prove, we show that $\boxed{\frac{d}{dt} f = 0}$ $\quad \frac{d}{dt}\left(x + \frac{1}{3} y^2\right) = x' + \frac{2}{3} y y' =$

filling in ... $x + y^2 + \frac{2}{3} y(-y) = x + y^2 - \frac{2}{3} \ldots$ at time t always 0

★ $\frac{d}{dt}\begin{pmatrix}x\\y\end{pmatrix} = F\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}\overbrace{x^2}^{\text{always positive}}\\-y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ to find equilibrium

$\Rightarrow \begin{pmatrix}0\\0\end{pmatrix}$ is the only equilibrium

linearisation: $DF_{\begin{pmatrix}0\\0\end{pmatrix}} = \begin{pmatrix}2x & 0\\0 & -1\end{pmatrix}\Big|_{\begin{pmatrix}0\\0\end{pmatrix}} = \begin{pmatrix}0 & 0\\0 & -1\end{pmatrix} \rightsquigarrow \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}0\\-y\end{pmatrix}$

↳ x is not changing

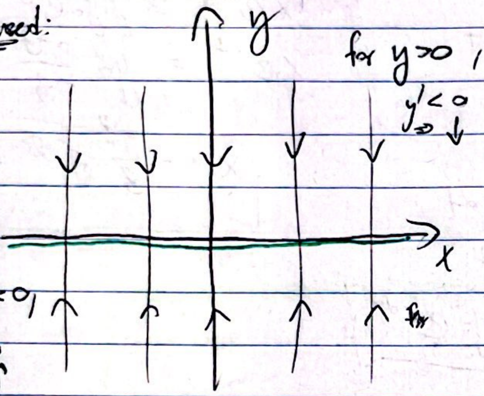
$\Rightarrow$ sols evolve along straight ~~lines~~ vertical lines

Phase portraits:

linearised:

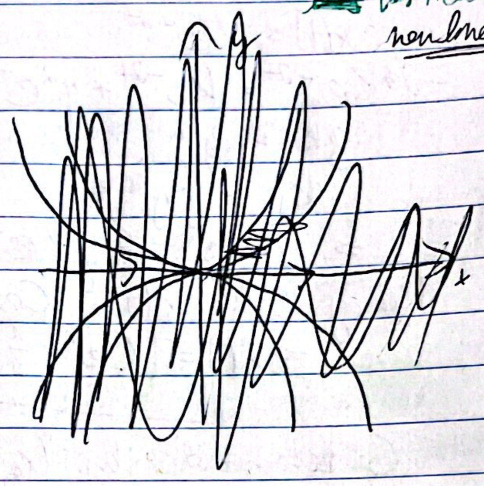
for $y>0$, $y'<0 \Rightarrow \downarrow$

whole x-axis is an equilibrium

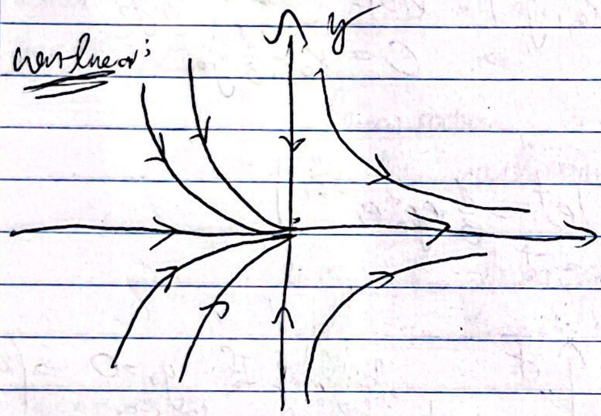


for $y<0$, $y'>0 \Rightarrow \uparrow$

nonlinear



nonlinear:



$x \rightsquigarrow$ always positive flow in x-direction

$\rightsquigarrow$ for $y<0$, positive flow
for $y>0$, negative flow

$\rightsquigarrow$ x-component ~~increases~~ approaches 0 faster than y-comp because quadratic in x

↳ dominated by slow dynamics: as before, approach along ~~the~~ smaller $|\lambda_i|$

Even if we look close, the linearisation is not a good approximation because the phase portraits are very different (also near the equilibrium)

$\Rightarrow$ this is because of the line of equilibria in the linearisation case

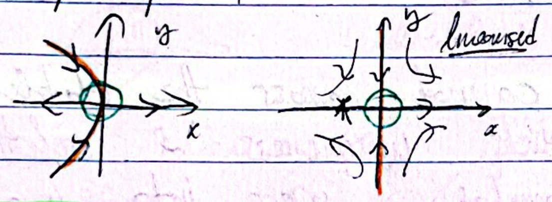
Linearization Th^m = Hartman - Grobman Th^m:

Suppose the system $\vec{X}' = F(\vec{X})$ has an equilibrium point $\vec{X}_0$ which is hyperbolic (i.e. $DF(\vec{X}_0)$ is hyperbolic - $\forall \mathrm{Re}(\lambda) \neq 0$). Then the flow of the linearised system is topologically conjugate to the flow of the original system in a neighborhood of $\vec{X}_0$ (locally).

↳ linearisation is a good approximation if DF is hyperbolic ($\forall \lambda : \mathrm{Re}(\lambda) \neq 0$)

~~[illegible]~~ ☆ example p. 31, 32

original



If we have a saddle, the linearised system has a stable line tangent to a stable curve

Stable Curve Thm:

Suppose the system: $x' = \lambda x + f_1(x,y)$

$y' = -\mu y + f_2(x,y)$

satisfies $-\mu < 0 < \lambda$ ($\Rightarrow$ saddle)

and $\dfrac{f_j(x,y)}{\sqrt{x^2+y^2}} \xrightarrow[(x,y)\to(0,0)]{} 0$

*for the linearised system

(meaning: f_1, f_2 are of order at least x^2, y^2.)

$\to$ y-axis is the stable curve* (approaches 0 because $-\mu < 0$)

Thm $\exists \varepsilon > 0$, and a curve $x = h^s(y)$ is defined for $|y| < \varepsilon$,

s.t. $x = h^s(y=0) = 0$ (curve passes through equilibrium)

and following properties hold

holds locally

1. The curve is invariant and solutions with initial cond. on the curve approach $\binom{x}{y} = \binom{0}{0}$ as $t \to \infty$
2. The curve $x = h^s(y)$ passes through the origin tangentially to the y-axis (= stable curve of the linearised system)
3. All other solutions with initial cond. $\notin$ curve $h^s(y)$ but in a disk of radius ε centered at the origin eventually leave the disk as t increases.

Then $x = h^s(y)$ is called the local stable curve.

locally / at least

- the local stable curve is given by a graph over the y-axis (which is the stable curve of the linearized system)
- there is an analogous thm for the local unstable curve
- the local stable curve can exist more globally (but generally, not necessarily as a graph anymore)
- without restriction we can assume canonical form because we can apply a linear transformation to any system with a saddle to bring it to the canonical form - this will not alter the saddle and it will not ~~...~~ add/remove higher order terms

- solutions cannot cross the stable curve (contradicts uniqueness of solutions)
  $\Rightarrow$ separates space into two parts (also holds for higher dimensional manifolds)
- intersection of stable & unstable manifolds leads to chaos

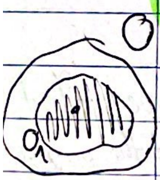
Stability of equilibria

Def: $x' = F(x)$. Let $\bar{X}^*$ be an equilibrium, i.e. $F(\bar{X}^*) = 0$.

- $\bar{X}^*$ is **stable** if $\forall$ open neighborhood O of $\bar{X}^*$, $\exists$ another open neighborhood O_1 of $\bar{X}^*$, s.t. $\forall$ solutions with $\bar{X}(0) := \bar{X}_0 \in O_1$ $\Rightarrow$ it follows that $\bar{X}(t) \in O$ $\forall t \geq 0$.

Lyapunov stable

- $\bar{X}^*$ is asymptotically stable if it is stable and the neighborhood O_1 of $\bar{X}^*$ can be chosen such that $\bar{X}(t) \to \bar{X}^*$ as $t \to \infty$ if $\bar{X}_0 \in O_1$.



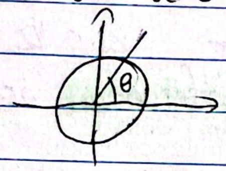
↳ stable: given open neigh. O, we can find another O_1 s.t. if we begin in O_1, sol. stays in O.

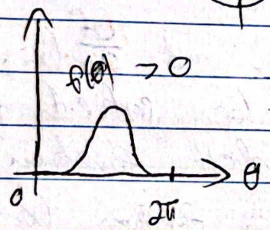
↳ asympt.: we can choose O_1 s.t. the solution will converge to $\bar{X}^*$

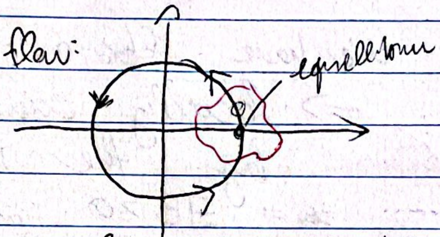
- asymptotic stability $\Rightarrow$ Lyapunov stability
- ☆ center equilibria are stable but not asymptotically stable
- ☆ for hyperbolic equilibria (Hartman – Grobman applies), we can conclude stability from linearisation
  - if $Re(\lambda) < 0 \ \forall \lambda$, ~~then~~ $\Rightarrow$ asymptotic stability
  - if at least one $\exists \lambda$ has $Re(\lambda) > 0 \Rightarrow$ not stable
- the condition "... is stable and ..." in def of asymptotic stability cannot be omitted:

☆ system on a circle: $\theta' = \sin^2 \frac{\theta}{2}$, $\theta \in [0, 2\pi] \in \mathbb{R}/2\pi\mathbb{Z}$

well-defined because 2π-periodic ✓







- all solutions will converge to $\theta = 0$ but the point is not stable because if given neighbourhood as shown, it ~~will~~ might first leave and eventually ~~look~~ come back

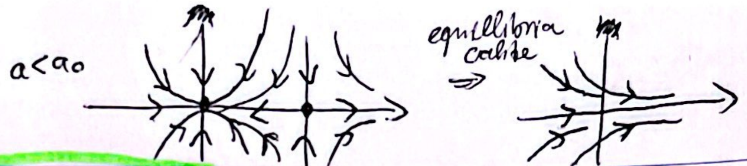
Bifurcations of equilibria

- consider family of systems $\bar{x}' = F(\bar{x}, a)$, $\bar{x} \in \mathbb{R}^n$, $a \in \mathbb{R}^m$ parameters

~~difference~~

- bifurcation = solution structure (where equilibria and their types) changes under variation of a.

- Consider 1D system depending on a single parameter:
  $x' = f(x, a)$, $x \in \mathbb{R}$, $a \in \mathbb{R}$



sink

$\parallel$

- **saddle-node bifurcation** occurs at $a = a_0$,
  if $\exists \varepsilon > 0$ s.t.
  1. if $a_0 - \varepsilon < a < a_0$, the system has 2 equilibria
  2. if $a = a_0$, the system has 1 equilibrium
  3. if $a_0 < a < a_0 + \varepsilon$, the system has no equilibrium

Thm: Suppose for $x' = f(x,a)$, $f \in C^\infty$, and if
1. $f(x_0, a_0) = 0$
2. not hyperbolic $\Rightarrow$ $\frac{\partial f}{\partial x}(x_0, a_0) = 0$ } not hyperbolic equilibrium at x_0 for a_0
3. $\frac{\partial^2 f}{\partial x^2}(x_0, a_0) \neq 0$
4. $\frac{\partial f}{\partial a}(x_0, a_0) \neq 0$ for $x_0 \in \mathbb{R}$, $a_0 \in \mathbb{R}$.

Thm a saddle-node bifurcation occurs at $a = a_0$.

important proof

Proof: We have $f(x_0, a_0) = 0$ and $\frac{\partial f}{\partial a}(x_0, a_0) \neq 0$

$\Rightarrow$ locally can be solved for a by the implicit function thm

(part of) Implicit function Theorem

$= \exists \varepsilon > 0$, and a function $(x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$ s.t. $f(x, a(x)) = 0$ and $a(x_0) = a_0$ $\quad x \mapsto a(x)$

(and more)

we have $\boxed{a'(x_0) = -\dfrac{\frac{\partial f}{\partial x}(x_0, a_0)}{\frac{\partial f}{\partial a}(x_0, a_0)}} = 0$ because $\frac{\partial f}{\partial x} = 0$ (not hyperbolic)

$$\frac{df}{dx}(x, a(x)) = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial a}\frac{da}{dx} = 0 \Rightarrow \frac{\partial f}{\partial x} = -\frac{\partial f}{\partial a} a'$$

$$\Rightarrow a' = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial a}}$$

$$a''(x_0) = \frac{d}{dx} a' = -\frac{d}{dx} \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial a}}\Bigg|_{x = x_0} =$$

$$= -\frac{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial a}\cancel{a'}\right)\frac{\partial f}{\partial a} - \cancel{\frac{\partial f}{\partial x}}\left(\frac{\partial^2 f}{\partial x \partial a} + \frac{\partial^2 f}{\partial a^2}\cancel{a'}\right)}{\left(\frac{\partial f}{\partial a}\right)^2}\Bigg|_{x = x_0}$$

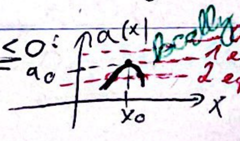
$a'(x_0) = 0$

$\frac{\partial f}{\partial x}(x_0) = 0$

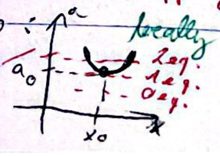
$$= -\frac{\frac{\partial^2 f}{\partial x^2}}{\frac{\partial f}{\partial a}}\Bigg|_{x = x_0} \neq 0 \text{ by condition 3.}$$

see p. 4 for details

hence the graph of $a(x)$ near x_0 looks like:

$a''(x_0) < 0$: 

with opposite stabilities

$a''(x_0) > 0$: 

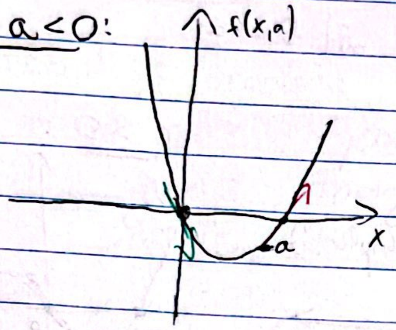
$\square$

Scanned with CamScanner

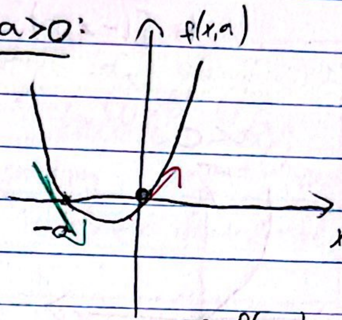
• further bifurcations: transcritical bifurcation

☆ $x' = f(x,a) = ax \pm x^2 = 0$ – here we take $\ldots + x^2$ but $-x^2$ is similar

fixed $a<0$:

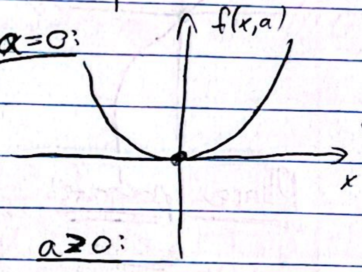


fixed $a>0$:

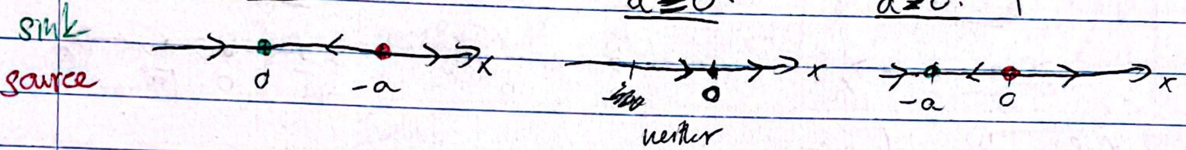


• equilibrium at $x=0$ and $x=-a$

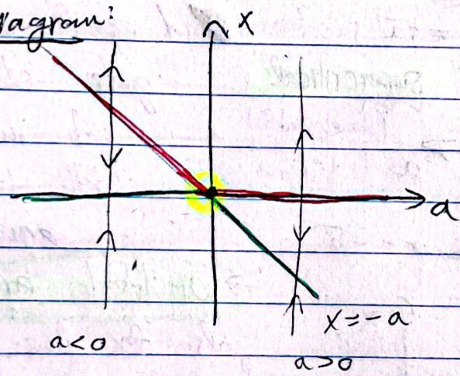
$a=0$:



Phase portraits:



Bifurcation diagram:



~~2 equilibria~~

Here bifurcation at $a = a_0 = 0$

$\frac{\partial f}{\partial a}(x_0,a_0) = \frac{\partial f}{\partial a}(0,0) = 0$

2 equilibria of opposite stability which cross and exchange their stability = transcritical bifurcation

Thm: Suppose for $x' = f(x,a)$, $f \in C^\infty$. If

1.+2.: non-hyperbolic equilibrium: $f(x_0,a_0)=0$, $\frac{\partial f}{\partial x}(x_0,a_0)=0$

3.: $\frac{\partial f}{\partial a}(x_0,a_0)=0$

4.: $\frac{\partial^2 f}{\partial x \partial a}(x_0,a_0) \neq 0$

5.: $\frac{\partial^2 f}{\partial x^2}(x_0,a_0) \neq 0$

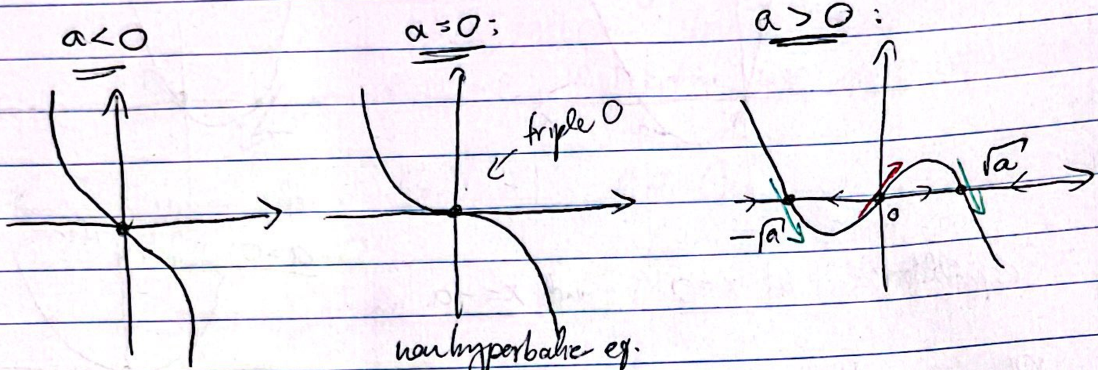
then we have a transcritical bifurcation at $a=a_0$

Proof: similar as for prev. Thm but now we look at 2 implicit fun's. Take $f(x,a) = (x-x_0)\,g(x,a)$, then apply implicit fun. Thm on $g(x,a)$.

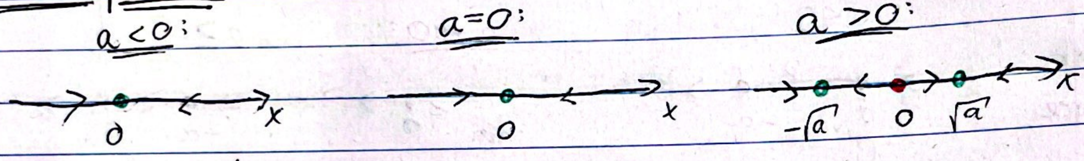
Pitchfork Bifurcation

$x' = f(x,a) = ax \mp x^3 = x(a \mp x^2)$ (considering $\ominus$ but $\oplus$ is similar)

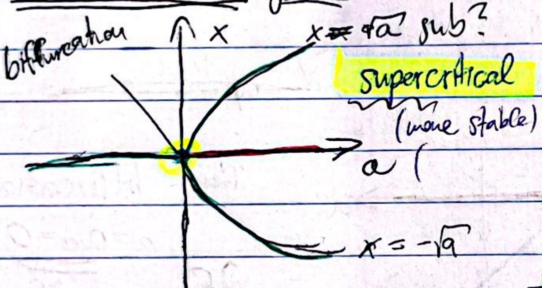
$\hookrightarrow f(-x,a) = -f(x,a)$ odd function



Phase portraits:



Bifurcation diagram:

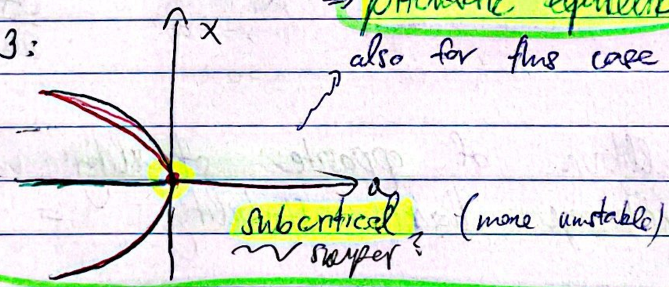


1 equilibrium (sink) gets changed (into source) and in doing so gives birth to two new equilibria

$\Rightarrow$ pitchfork equilibrium

also for this case

for $f = ax + x^3$:



Thm: Suppose $x' = f(x,a)$, $f \in C^\infty$: if

1.+2. non-hyperbolic equilibrium (at bifurcation)

3. $\frac{\partial f}{\partial a}(x_0, a_0) = 0$
4. $\frac{\partial^2 f}{\partial x^2}(x_0, a_0) = 0$
5. $\frac{\partial^2 f}{\partial x \partial a}(x_0, a_0) \neq 0$
6. $\frac{\partial^3 f}{\partial x^3}(x_0, a_0) \neq 0$ — sign decides super <0 / subcritical >0

for $x_0 \in \mathbb{R}$, $a_0 \in \mathbb{R}$.

Then a pitchfork bifurcation occurs at $a = a_0$.

Hopf bifurcation

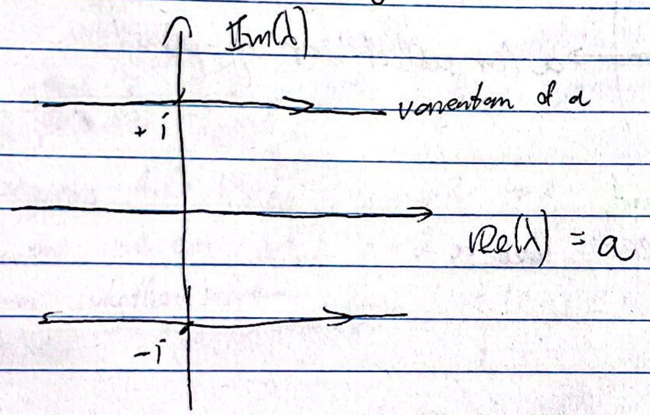
only in 2D+

- previously: breaking hyperbolicity by going through 0.
  ⇒ but also possible to break by going through imaginary axis ⇒ only possible for planar or more dimensional systems

$$\begin{cases} x' = ax - y - x(x^2+y^2) \\ y' = x + ay - y(x^2+y^2) \end{cases} \Rightarrow \text{equilibrium at } (0,0)$$

↳ linearisation at $(0,0)$:

$$\vec{x}' = \begin{pmatrix} a - 3x^2 - y^2 & -1-2xy \\ 1-2xy & a - 3y^2 - x^2 \end{pmatrix}\Bigg|_{\binom{0}{0}} \vec{x} = \underbrace{\begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}}_{\text{canonical form of complex evals}} \vec{x}$$



$$\boxed{\lambda_{1,2} = a \pm i}$$

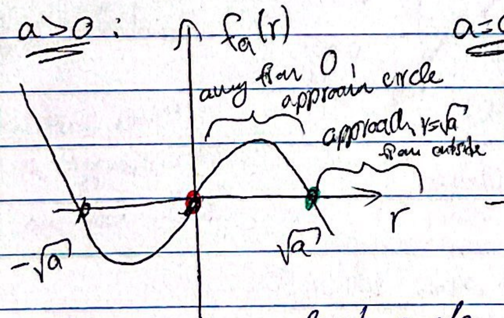
in terms of polar coordinates:

$$\text{steady increase} - \begin{cases} r' = ar - r^3 = f_a(r) = r(a-r^2) \\ \theta' = 1 \end{cases}$$

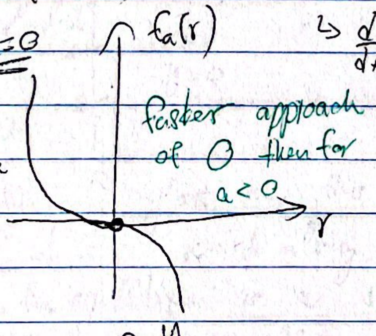
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \quad r = \sqrt{x^2+y^2}$$

↳ $\frac{d}{dt}r$ and substitute x' and y' from the system above

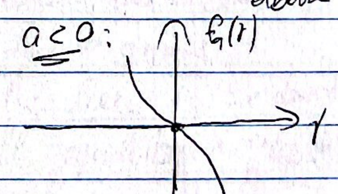
$a>0$:

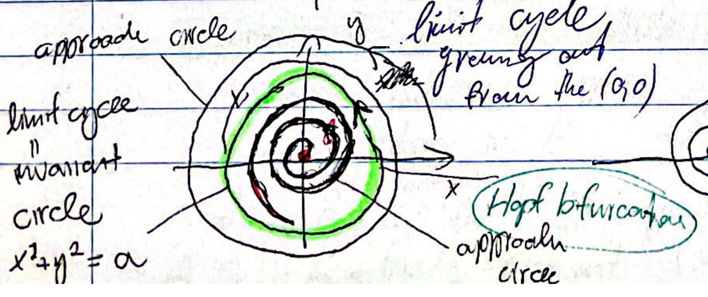


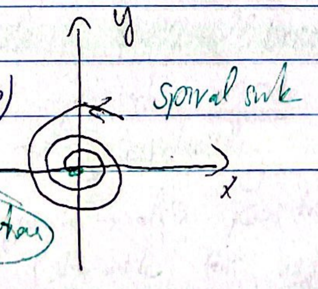
$a=0$

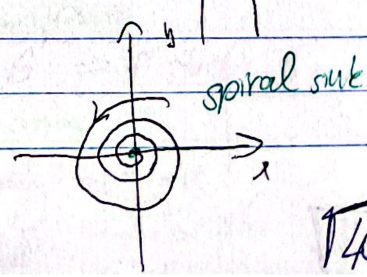


$a<0$:



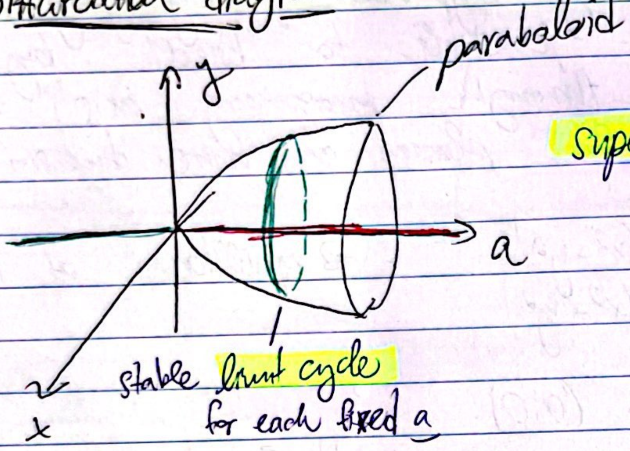






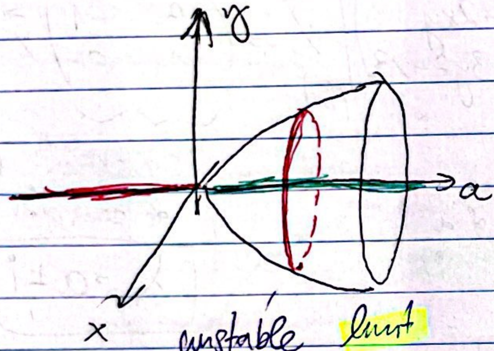
→ Hopf bifurcation

Bifurcation diagram



supercritical
↳ unstable after stable circle is born

stable limit cycle
for each fixed a



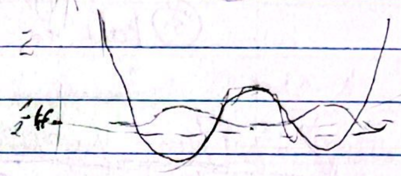
subcritical

a (or reflect over xy-plane)

unstable limit
cycle for each fixed a

Techniques to globally analyse non-linear systems

Nullclines = zero level set of one of f_i's



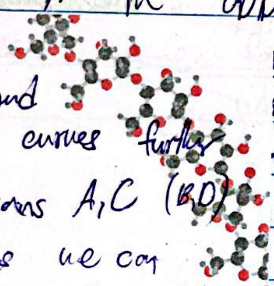
$$\begin{cases} x_1' = f_1(x_1, \dots, x_n) \\ \vdots \\ x_n' = f_n(x_1, \dots, x_n) \end{cases}$$

x_j nullcline is the zero-level set of f_j: $\{\bar{x} \in \mathbb{R}^n : f_j(\bar{x}) = 0\}$

↳ $n-1$ dimensional surface

↳ on each side of nullcline, vector field pointing in opposite directions?

celulóza



stable and (unstable) curves further divide regions A, C (B, D)

⇒ from this we can determine whether solution

...rmally: nullclines divide $\mathbb{R}^n$ into open sets ...nich the vector field has a particular ...n.

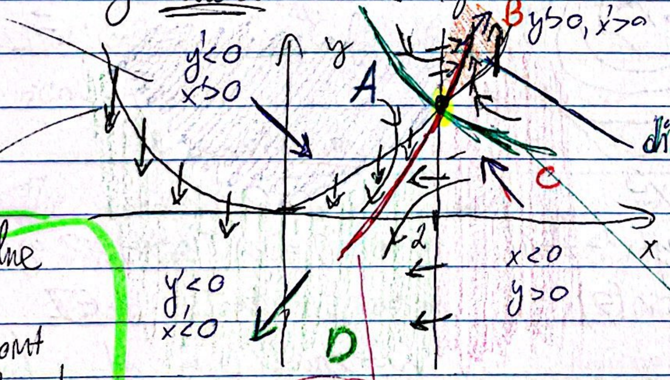
$y - x^2 = x'$

$x - 2 = y'$

...cline: $\{(x,y) \in \mathbb{R}^2 \mid y - x^2 = 0\}$

y-nullcline: $\{(x,y) \in \mathbb{R}^2 \mid x - 2 = 0\}$

above parabola $y > x^2$ ⇒ arrows in positive x-direction on line



increasing time

If start in A/C, eventually we ~~eventue~~ leave in B/D. Which?

direction of arrows determined by the other eq: i.e. $x' = y - x^2$

stable curve

locally from lin., globally??

x-nullcline ⇓ arrows point in $\pm y$-direction on the line

unstable curve

$y' = 0$ on this line ⇒ arrows point along the line (in $\pm x$ direction)

in this case, the equilibrium is a saddle equilibrium ⇒ (un)stable lines

- intersection of all nullclines = equilibrium point

B: both nullcline arrows point inside B ⇒ if start in B, we stay in B (for increasing t) ⇒ positively invariant

⇒ similarly for D region → also positively invariant

↳ A, C – negatively invariant (stay inside for negative time direction)

Stability of equilibria and basins of attraction

- if hyperbolic ⇒ stability from linearisation

(?) how to determine stability of a non-hyperbolic ~~equilibrium~~ equilibrium?

for hyperbolic, determine from linearisation

Thm: Lyapunov - stability of non hyperbolic eq.

Let $\vec{x}^*$ be an equilibrium of $\vec{x}' = \vec{F}(\vec{x})$ (i.e. $F(\vec{x}^*) = 0$).
Suppose for an open set U containing $\vec{x}^*$ ($\vec{x}^* \in U$), there's a C^1 function $L: U \to \mathbb{R}$ with

a) $L(\vec{x}^*) = 0$ and $L(\vec{x}) > 0$, $\forall \vec{x} \in U \setminus \{\vec{x}^*\}$

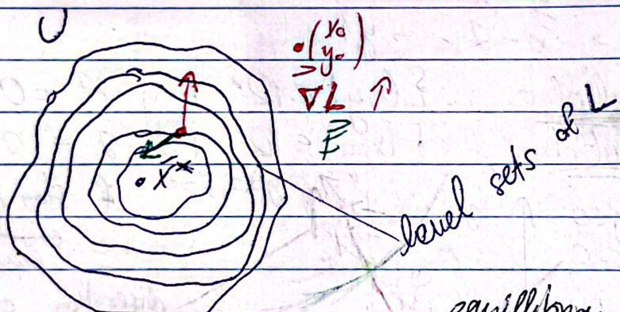
b) $\dot{L} := \vec{\nabla} L \cdot \vec{F} \leq 0$ in $U \setminus \{\vec{x}^*\}$

Then $\vec{x}^*$ is stable. L = Lyapunov Function

If instead of condition b) we have

c) $\dot{L} < 0$ in $U \setminus \{\vec{x}^*\}$ then $\vec{x}^*$ is asymptotically stable and U belongs to the basin of attraction of $\vec{x}^*$, which is the set of initial conditions for solution, that converge to $\vec{x}^*$ as $t \to \infty$. Then L = Strict Lyapunov function

L not increasing along $\vec{F}$.



$\nabla L \cdot F \leq 0$

⇒ angle between $\vec{\nabla} L$ and $\vec{F}$ is larger than $90°$ (or equal)

☆
$$x' = -x^3$$
$$y' = -y(x^2 + z^2 + 1)$$
$$z' = -\sin(z)$$

equilibria

⇒ $x = 0$

→ $0 = y(\pi k + 1)$ ⇒ $y = 0$

→ $z = \pi k$, $k \in \mathbb{Z}$

⇝ equilibria at $\begin{pmatrix} 0 \\ 0 \\ \pi k \end{pmatrix}$

Note: the planes $\{z = n\pi\}$ are invariant

- For example: solutions with initial condition $|z| < \pi$, remain in layer $z \in (-\pi, +\pi)$, i.e. bounded by planes $\{z = \pi\}$ and $\{z = -\pi\}$

Notation:
{condition} ⇒
⇒ all points satisfying given condition, e.g. $\{z = \pi\}$

Consider the equilibrium $\begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. It has linearisation

$$\vec{X}' = \begin{pmatrix} -3x^2 & 0 & 0 \\ -2xy & -(x^2+z-1) & -2yz \\ 0 & 0 & -\cos(z) \end{pmatrix}\Bigg|_{\begin{pmatrix}0\\0\\0\end{pmatrix}} \quad \vec{Y}' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \vec{X}$$

Eigenvalues: $0, -1, -1$

$\Rightarrow$ not hyperbolic equilib.

$\Rightarrow$ linearisation doesn't tell us anything (namely anything about the stability)

Set $L(x,y,z) = x^2+y^2+z^2$

$=$ square distance to $\begin{pmatrix} x^* \\ y^* \\ z^* \end{pmatrix}$

When constructing Lyapunov function, always first try $L = (\text{distance to } \vec{X}^*)^2$

Now check if satisfies. Then:

② domain of def? at most the layer between $\{z=-\pi\}$ and $\{z=\pi\}$ because $-\pi$ and π are equilibria and solutions cannot cross over each other.

$$\mathring{L}(x,y,z) = \underbrace{L_x}_{\frac{\partial L}{\partial x}}(-x^3) + \underbrace{L_y}_{\frac{\partial L}{\partial y}}(-y(x^2+z-1)) + \underbrace{L_z}_{\frac{\partial L}{\partial z}}(-\sin z)$$

$$= \underbrace{-2x^4}_{\le 0} \underbrace{-2y^2 \underbrace{(x^2+z^2-1)}_{>0}}_{\le 0} \underbrace{- 2z\sin(z)}_{\le 0}$$

on this layer: $0<z<\pi$: $\sin(z)>0$, $z>0$

$-\pi<z<0$: $\sin(z)<0$, $z<0$

$\Rightarrow$ at least a Lyapunov function.

$\Rightarrow$ can this be 0? $\Rightarrow$ only at $\vec{X}^* = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$\Rightarrow$ a strictly Lyapunov function on layer $\{-\pi<z<\pi\}$

$\Rightarrow$ The whole layer belongs to the basin of attraction. In fact the layer is the full basin of attraction.

• if there are no sols entirely along L-level sets in P

$\Rightarrow$ asymptotic stability for nonstrict L-fun.

Thm: Lasalle's Invariance principle

Let $\vec{X}^*$ be an equilibrium of $\vec{X}' = F(\vec{X})$.

Let U be an open neighbourhood of $\vec{X}^*$, $L: U \to \mathbb{R}$ Lyapunov fun. (doesn't need to be strict)

$P \subset U$, $\vec{X}^* \in P$, P closed and bounded ($\Rightarrow$ compact)

and positively invariant, there are no entire sols in $P \setminus \{\vec{X}^*\}$ on which L is constant (live on level set of L).

Then $\vec{X}^*$ is asymptotically stable and P belongs to the basin of attraction.

Proof: By contradiction. Let $\bar{X}(t)$ be a solution with $\bar{X}(t) \in P \quad \forall t \geq 0$

($\bar{X}(0) \in P$ but P is positively invariant $\Rightarrow \forall t \geq 0$)

Suppose $\bar{X}(t) \underset{\text{doesn't converge to}}{\not\longrightarrow} \bar{X}^*$ as $t \to \infty$

Consider sequence $\{X(t)\}, t \in \mathbb{N}$. Because this sequence is contained in ~~the~~ a compact set, it has accumulation point(s) $= Z$.

P compact $\Rightarrow \exists$ sequence of times (t_n), $t_n \to \infty$ as $n \to \infty$ and $\exists Z \in P \setminus \{\bar{X}^*\}$ with $\bar{X}(t_n) \to Z$

Show: i) $\phi_t(Z) \in P \quad \forall t \in \mathbb{R}$
ii) L constant on $\phi_t(Z)$

} this will be the contradiction

(i) • $\phi_t(Z) \in P \quad \forall t \geq 0$ because $\phi_0(Z) \in P$ and P is positively invariant.

• $\phi_t(X(t_n)) \in P \quad \forall t \in [-t_n, 0]$
$= \phi_{t+t_n}(X(0)) \in P \Rightarrow$

• $\phi_t(X(\underbrace{t_{n+k}}_{\geq t_n})) \in P, \quad \forall t \in [-t_n, 0], \forall k \in \mathbb{Z}_{\geq 0}$

• $X(t_{n+k})$ can be viewed as a sequence in k and it also converges to Z as $k \to \infty$.

$\Rightarrow$ By continuity of the flow, $\underbrace{\phi_t(X(t_{n+k}))}_{\in P} \xrightarrow[k\to\infty]{} \underbrace{\phi_t(Z)}_{\in P}$

$\forall t \in [-t_n, 0] \quad \in P \Rightarrow \in P$

As $n \to \infty$, $t_n \to \infty \Rightarrow -t_n \to -\infty$

$\Rightarrow \phi_t(Z) \in P, \forall t \in \mathbb{R}$.

(ii) Let $\alpha := L(Z) \Rightarrow L(\bar{X}(t_n)) \geq \alpha, \forall n \geq 0$
and $L(\bar{X}(t_n)) \to \alpha$ as $n \to \infty$ by continuity of L.

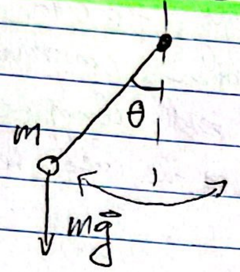
Let $s \in \mathbb{R} \Rightarrow X(t_n + s) = \phi_s(X(t_n)) \to \phi_s(Z)$

Also $L(\bar{X}(\underbrace{t_n+s}))\to \alpha$ as $n \to \infty$
a new sequence & $\Rightarrow$ (?) convergence to Z

but L is nonincreasing along $F \Rightarrow X(t_n+s)$ is contained between $X(t_n)$ and we get a subsequence along which L is nonincreasing

$\Rightarrow L(\phi_s(Z)) = \alpha \quad \forall s \in \mathbb{R}$ $\Rightarrow$ contradiction because we get a sol. constant along L-level sets $\square$

★ Damped Pendulum



damping factor

$$ml\,\frac{d^2\theta}{dt^2} = -mg\sin\theta - b\,\underbrace{l\,\frac{d\theta}{dt}}_{\text{velocity of pendulum}}$$

$$\frac{l}{g}\,\frac{d^2\theta}{dt^2} = -\frac{bl}{mg}\,\frac{d\theta}{dt} - \sin\theta$$

$[s^2]\,[s^{-2}]$

⇒ introduce new ~~time~~ dimensionless time $\tau = \sqrt{\frac{g}{l}}\,t$

$[\varnothing]\quad [s^{-1}][s]$

$$\Rightarrow \frac{d^2\theta}{d\tau^2} = -\underbrace{\frac{bl}{mg}\sqrt{\frac{g}{l}}}\,\frac{d\theta}{d\tau} - \sin\theta$$

dimensionless ⇒ $\boxed{\frac{b}{m}\sqrt{\frac{l}{g}} \equiv \tilde{b}}$

$$\Rightarrow \boxed{\frac{d^2\theta}{d\tau^2} = -\tilde{b}\,\frac{d\theta}{d\tau} - \sin\theta}$$ equation of motion

1st order system
$$\begin{cases} \dfrac{d\theta}{d\tau} = v \\[2mm] \dfrac{dv}{d\tau} = -\tilde{b}v - \sin\theta \end{cases}$$

⇒ equilibrium $v = 0$, $\theta = n\pi$ $(n \in \mathbb{Z})$

but θ on a circle ~~and~~ $\theta \in [0, 2\pi)$

⇒ two equilibria: $0, \pi$

⇒ equilibria: $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\pi\\0\end{pmatrix}$

hanging position — upward position

Stability?

First try to linearise: $(\theta, v) = (0,0)$, $Y = \begin{pmatrix}\Delta\theta\\ \Delta v\end{pmatrix}$

$$Y' = DF_{\binom{0}{0}}\,Y$$

$$DF_{(\theta)} \Rightarrow \begin{pmatrix} 0 & 1 \\ -\cos\theta & -\tilde{b}\end{pmatrix}\Bigg|_{\binom{0}{0}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & -\tilde{b}\end{pmatrix}}_{A}$$

$$\lambda_{1,2} = \tfrac{1}{2}\left(\operatorname{tr}A \pm \sqrt{(\operatorname{tr}A)^2 - 4\det(A)}\right) = \tfrac{1}{2}\left(-\tilde{b} \pm \sqrt{\tilde{b}^2 - 4}\right).$$

For $\tilde{b} > 0$ (friction) ⇒ $\lambda_{1,2}$ have negative real parts

⇒ the equilibrium is asymptotically stable

Now use Prenan Thm to study basin of attraction →

for physical systems, useful Lyapunov function could be the energy function.

$E(\theta,v) := \frac{1}{2}v^2 + 1 - \cos\theta$

= function conserved in time along solutions (not the case here: friction)

$\dot{E} = \nabla_{(\theta,v)} E \cdot F$, $F = \begin{pmatrix} v \\ -\tilde{b}v - \sin\theta \end{pmatrix}$

$\dot{E} = \begin{pmatrix} \sin\theta \\ v \end{pmatrix} \cdot \begin{pmatrix} v \\ -\tilde{b}v - \sin\theta \end{pmatrix} = v\sin\theta - \tilde{b}v^2 - v\sin\theta = -\tilde{b}v^2 \le 0$ ✓

(indeed if $b = 0$, then E is conserved along solution curves ($\dot{E} = 0$))

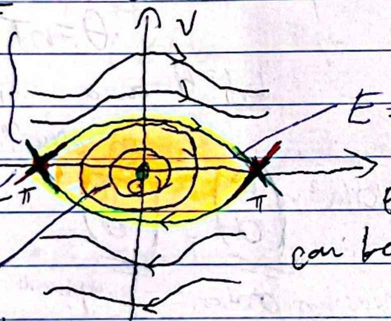
E is indeed a Lyapunov function (after checking $E(0,0) = 0 + 1 - 1 = 0$ ✓, $E \neq 0$ away from $(0,0)$)

$\binom{0}{0}$ is stable by Lyapunov theory

? asymptotic stability

? basin of attraction

for $\tilde{b} = 0$: E is conserved ⇒ for phase portrait consider level sets of energy function



$E > 2$ ⇒ → rotation (clockwise or counterclockwise)

$E = 2$ ↳ between oscillation and rotation ↳ if start with $E = 2$, in $t \to \infty$, we reach the equilibrium

can be periodically extended

separatrix (separation rotation and oscillation)

saddle points of E and of dynamics

for small displacement/V ⇒ oscillations

↳ stable & unstable lines coincide

- Energy cannot increase (for any $\tilde{b} \ge 0$), therefore when we begin inside some level set, we can never cross it going out.

- Let $0 < c < 2$ and $P_c := \{(\theta, v) \in [-\pi, \pi] \times \mathbb{R} \mid E(\theta, v) \le c\}$

$\tilde{b} > 0$: Because energy cannot increase, P_c is positively invariant and compact.

To show: There is no solution entirely contained in P_c, apart from the equilibrium on which E is constant if $\delta > 0$.

Suppose $(\theta(t), v(t))$ with $(\theta(0), v(0)) \in P_c$ has ~~(for contradiction)~~

$E(\theta(t), v(t)) = \text{const.} \Rightarrow \dot{E}(\theta(t), v(t)) \equiv 0 \;\; \forall t$

$\Rightarrow \; -\delta v^2(t) \equiv 0 \;\; \forall t$

$\Rightarrow \; v(t) = 0 \;\; \forall t$ and $\sin\theta(t) \equiv 0 \;\; \forall t$

from $\dot{v} = -\delta v - \sin\theta = 0$ $\quad \theta(t) \equiv 0 \;\; \forall t$

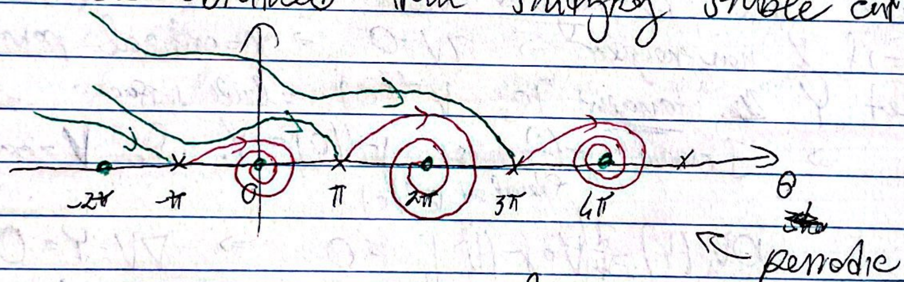
$\Rightarrow (\theta(t), v(t)) = (0,0) \;\; \forall t$ but this is the equilibrium sol.

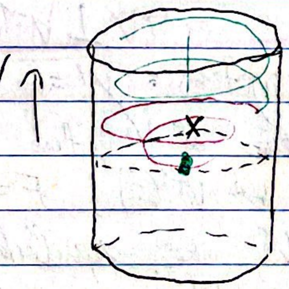
[Holds for all P_c's $\Rightarrow$ their union belongs to the basin of attraction

By Lasalle's Invariance Principle, $(\theta, v) = (0,0)$ is asymptotically stable and P_c belongs to the basin of attraction for any $0 < c < 2$.

(In reality full basin of attraction is the whole plane (but separatrix) but we cannot get that here)
can be obtained from studying stable curves

$\delta > 0$:





we can visualise as a cylinder

• next: Thm to rule out chaotic sols for planar systems (need 3D+, or a curved surface e.g. torus can have chaos) by knowing the limit / accumulation points are simple

Gradient Systems : $\boxed{X' = -\nabla V(X)}$

potential function

- simple flows
- $DV_x(\vec{Y}) = \vec{\nabla} V(x) \cdot \vec{Y}$
- let $X(0) = Y_0$, $\dot{V} : \mathbb{R}^n \to \mathbb{R}$, i.e. $\dot{V}(X) = \frac{d V(X(t))}{dt}\Big|_{t=0}$

$\dot{V} = \frac{dV}{dt}$

X not equilibrium $\Rightarrow \dot{V}(x) < 0$.

Prop: V is strictly decreasing along nonconstant solutions of $X' = -\nabla V(X)$. Moreover, $\dot{V}(X) = 0 \Leftrightarrow X$ - equilibrium.

$\boxed{\dot{V} = \vec{\nabla} V \cdot \vec{F}}$

Proof: $\dot{V}(x) \overset{\vec{\nabla V} \cdot \vec{F}}{=} DV_x(X') = \nabla V \cdot (-\nabla V) = -|\nabla V|^2 \leq 0$.

$\dot{V}(X) = 0 \overset{\nabla V}{\Leftrightarrow} \nabla V = 0 \Leftrightarrow X' = 0 \Leftrightarrow X^*$ equilibrium $\square$

If X^* = isolated (critical point) minimum of $V \Rightarrow X^*$ asymptot. stable

$\Downarrow$

$\dot{V}(x) < 0$ around it

- Level surfaces (dim $n-1$): $\frac{\partial V}{\partial x_n} \neq 0 \xrightarrow{\text{Implicit fn Thm } \exists g} V(x_1, \dots, x_{n-1}, g(x_1, \dots, x_{n-1})) = c$

for $n = 2 \Rightarrow$ curves

$\Rightarrow$ near X, $V^{-1}(c)$ looks like graph of g

- if $V^{-1}(c)$ not passing through equilibrium (only regular points: $\nabla V \neq 0$), then c = regular value
- if X non-regular $\Rightarrow \nabla V = 0 \Rightarrow X$ = critical point

Thm

$Z = \alpha$ or ω limit pnt of $X' = -\nabla V$ $\Rightarrow Z$ is an equilibrium

- Let Y tangent to $V^{-1}(c)$ = level surface, at X

$\Rightarrow \exists$ curve $\gamma(t)$ s.t. $\gamma'(0) = Y$. Since V = const. along γ,

$\in$ level set $V^{-1}(c)$

$DV_x(Y) = \frac{d}{dt}(V \circ \gamma(t))\Big|_{t=0} = 0 \Rightarrow \nabla V \cdot Y = 0 \Rightarrow \nabla V \perp Y$

$\Rightarrow \nabla V \perp V^{-1}(c)$

Thm: $\Rightarrow$ Properties of gradient system: $X' = -\nabla V(X)$

1. If c is a regular value of V, then the vector field is perpendicular to level set $V^{-1}(c)$: $F = -\nabla V \perp V^{-1}(c)$
2. The critical points of V are equilibria of system
3. If critical point is an isolated minimum of V, then it is asymptotically stable equilibrium point.

?

Prop: Let Z be an α-limit point or an ω-limit point of a solution of a gradient flow. Then Z is an eq. point

symmetric matrix

Prop: For a gradient system $X' = -\nabla V(X)$, the linearized system at any equilibrium point has only real eigenvalues

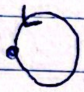
$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Hamiltonian Systems

$$\boxed{\begin{aligned} x' &= \frac{\partial H}{\partial y} \\ y' &= -\frac{\partial H}{\partial x} \end{aligned}} \qquad \boxed{\bar{X}' = J \nabla H(x)}$$

- $H: \mathbb{R}^2 \to \mathbb{R}^2$ = Hamiltonian function, C^∞
- H is a constant of motion $\Rightarrow$ const. along every solution curve
  $\Rightarrow \dot{H} = 0$

- Assume H nonconstant on any open set
  - $\Rightarrow$ plot level curves $H(x,y) = c$
  - $\Rightarrow$ solutions lie on these curves
  - $\Rightarrow$ we only need directions
- Equilibria at critical points of H
- Homoclinic orbits and solutions



  - ~~equilibrium to which~~ solutions converge in forward and backward time direction ($t \to \pm\infty$).
  - = solutions s.t. ↺ to same equilibrium

vs. Heteroclinic: converge forward to 1 equilibrium and backwards to another

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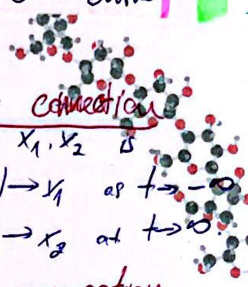


Heteroclinic connection between X_1, X_2 is if $\phi_t(X) \to X_1$ as $t \to -\infty$ and $\phi_t(X) \to X_2$ as $t \to \infty$

Homoclinic connection

Prop: Suppose (x_0, y_0) is an equilibrium point for a planar Hamiltonian system. Then the eigenvalues of the linearised system are either $\pm\lambda$ or $\pm i\lambda$, where $\lambda \in \mathbb{R}$.

Closed orbits & Limit sets

Limit sets

ω (∞)

$Y \in \mathbb{R}^n$ is an ω-limit point for a solution through X if $\exists$ sequence $t_n \to \infty$ s.t. $\lim_{n\to\infty} \phi_{t_n}(X) = Y$.

↳ solution curve through X accumulates on Y as $t \to \infty$

↳ set of all ω-limit points through X is ω-limit set X $\omega(X)$

→ similarly for α with $t \to -\infty$

α ($-\infty$)

$Y \in \mathbb{R}^n$ is an α-limit point for a sol. through X if $\exists$ sequence $t_n \to -\infty$ s.t. $\lim_{n\to\infty} \phi_{t_n}(X) = Y$

↳ limit set is invariant under ϕ

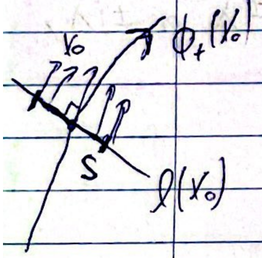
Prop: 1. If X and Z lie on the same solution, then $\omega(X)=\omega(Z)$ and $\alpha(X)=\alpha(Z)$.

2. If D is closed, positively invariant set and $Z\in D$, then $\omega(Z)\subset D$. and similarly for negatively invariant sets and $\alpha(Z)$: $\alpha(Z)\subset D_{neg}$

3. A closed and invariant set D and, in particular, a limit set, contains the α-limit and ω-limit sets of every point in it.
$\omega(D)\subset D$ - if D positively invariant
$\alpha(D)\subset D$ - if D negatively invariant

$X' = F(X)$

- $F(X_0)\neq 0$
- transverse line at X_0, denoted by $\ell(X_0)$ is straight line $\perp \vec{F}(X_0)$ passing through X_0. V_0 = unit vector at X_0, $\perp \vec{F}(X_0)$, then $h: \mathbb{R}\to \ell(X_0)$, $h(u) = X_0 + uV_0$
- $F(X)$ is continuous $\Rightarrow$ ~~we~~ $\vec{F}$ is not tangent to $\ell(X_0)$ in some open interval in $\ell(X_0)$ surrounding X_0. = local section at X_0 S.
  ↳ at each point of S, $\vec{F}$ points away from S, so solutions must cut across S. In particular $F(X)\neq 0$, $\forall X\in S$
- flow box in neighborhood of X_0
  ↳ description of behavior of flow in the neighborhood
  ↳ flow in flow box: $\underbrace{\text{parallel straight lines at const. speed}}_{\text{movement in}}$



- Construct $\Psi: N \longrightarrow \Psi(N) = V_0$ — flow box at X_0, $-|s|<\sigma$
  N: neighborhood of $(0,0)$ in $\mathbb{R}^2$; $\Psi(N)$: neighborhood of X_0
  $\Psi(s,u) = \phi_s(h(u))$, $\in\mathbb{R}^2$
  ↳ transverse line parametrisation
  if $X\in V_0$, then $\phi_t(X)\in S$ for unique $t\in(-\sigma,\sigma)$
  ↳ Ψ maps line $(0\ u)$ in N to the local section S
  ↳ Ψ maps horizontal lines in N to pieces of solution curves
  ↳ For small N, Ψ is injective
  ↳ $D\Psi$ (Jacobian) maps const. field $(1,0)$ in N to $F(X)$.

Prop: Let S be a local section at X_0 and suppose $\phi_{t_0}(Z_0)=X_0$. Let W be a neighborhood of Z_0. Then
$\Rightarrow$ $\exists$ open set $U\subset W$ containing Z_0 and $\exists$ cont. function $\tau: U\to\mathbb{R}$ s.t. $\tau(Z_0)=t_0$ and $\phi_{\tau(X)}(X)\in S$ $\forall X\in U$.

Prop: Let $X' = F(X)$ be a planar system and suppose that X_0 lies on a closed orbit γ. Let P be a Poincaré map defined on a neighborhood of X_0 in some local section. If $|P'(X_0)| < 1$, then γ is asymptotically stable.

- $X_0, X_1, \ldots \in \mathbb{R}^2$ sequence of distinct points on solution curve through X_0
  finite/infinite
  ⇝ monotone along the solution if $\phi_{t_n}(X_0) = X_n$ with $0 \leq t_1 < t_2 \ldots$ (similarly for other curves)

Prop: Let S be a local section for a planar system of differential equations and let $Y_0, Y_1, Y_2, \ldots$ be a sequence of distinct points in S that lie on the same solution curve. If this sequence is monotone along the solution, then it is also monotone along S.

Prop: For a planar system, suppose that $Y \in \omega(X)$. Then the solution through Y crosses any local section at no more than one point. Same is true if $Y \in \alpha(X)$.

Thm: Poincaré-Bendixson Thm
Suppose that Ω is a nonempty, closed, and bounded limit set of a planar system of differential equations that contains no equilibrium point. Then Ω is a closed orbit.

Alternative formulation:
$X' = F(X)$. $\mathcal{O} \subset \mathbb{R}^2$, positively invariant, open s.t. $\mathcal{O}$ contains finitely many equilibria.
For $X \in \mathcal{O}$, $\omega(X)$ is
- equilibrium point
- an equilibrium orbit (limit cycle) = closed orbit
- ~~connected~~ set equilibri with homoclinic/heteroclinic ~~conextions~~ connections

~~Def: x^*, fixed pt of $F(X)$~~

~~x^* is ω-limit point of x if $\exists \{t_n\}_{n\geq 0}^{\infty}$, $t_n \xrightarrow[n\to\infty]{} \infty$~~

~~s.t. $\phi_{t_n}(x) \xrightarrow[n\to\infty]{} x^*$~~

Corollary: if K is positively invariant + compact
$\Rightarrow$ either $\exists x^*$ = equilibrium point in K
or $\exists$ limit cycle = periodic orbit in K

Corollary: γ is periodic orbit and boundary of open positively invariant set $U \Rightarrow \exists$ equilibrium inside ($\exists x^* \in U$)

Thm: Liouville Thm
There is ~~no~~ A-stable (asymptotically stable) equilibrium point
$\Rightarrow$ no limit cycles ~~without equilibria~~

Corollary: (of Poincaré)

- Let H be a first integral of a planar system (Hamiltonian). If H is not constant on any open set, then there are no limit cycles.
- If L is a strict Lyapunov function for a planar system, then there are no limit cycles.

DISCRETE DYN. SYS.

$f: \mathbb{R} \to \mathbb{R}$, $x_0 \in \mathbb{R}$, C^∞

$x_1 = f(x_0)$

$x_2 = f(x_1)$

$\vdots$

$x_{k+1} = f(x_k) = f^k(x_0)$; $k = 0, 1, 2, \ldots$ ~ discrete time

$(x_0, x_1, x_2, \ldots) =$ orbit with "seed" x_0

Stationary solutions = fixed points of f

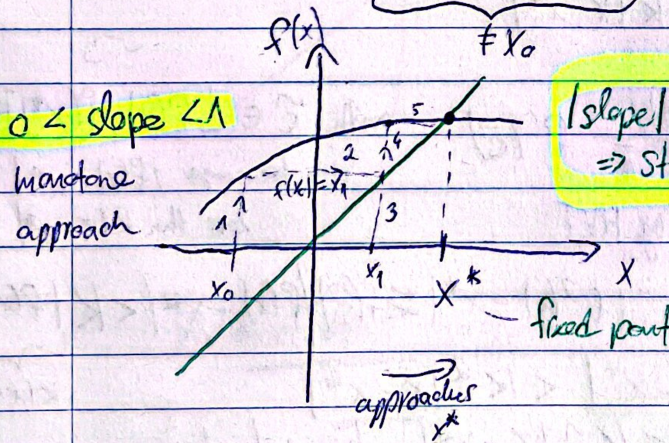
$$\boxed{x^* = f(x^*)}$$

Periodic points

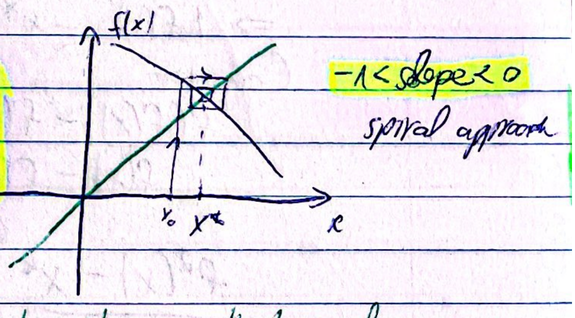
x_0 is called periodic of (minimal) period n if x_0 is a fixed point of f^n and not fixed point of f^k, $1 \le k < n$

↳ orbit of a periodic point is an n-cycle

$(x_0, \underbrace{x_1, \ldots, x_{n-1}}_{\neq x_0}, \underset{\overset{\shortparallel}{x_0}}{x_n}, x_{n+1}, \ldots)$



|slope| < 1 ⇒ stable



x^* — fixed point = intersection with diagonal $y = x$, $f(x) = x$

For |slope| > 1 ⇒ unstable < spiraling for slope < 0, monotone for slope > 0

Def: A fixed point x^* is called a sink or attracting if $\exists$ neighborhood U of x^* s.t. if $y_0 \in U$, then $f^k(y_0) \in U$ $\forall k$ and $f^k(y_0) \xrightarrow[k\to\infty]{} x^*$.
(stay in neighborhood and approach)
A fixed point x^* is called a source or repelling if $\exists$ neighborhood U of x^* s.t. $\forall y_0 \in U$, $\exists n \in \mathbb{Z}_{>0}$ s.t. $f^n(y_0) \notin U$.

Thm: x^* fixed point of $x_{k+1} = f(x_k)$, $k = 0,1,2,\dots$
Then:
- $|f'(x^*)| < 1 \Rightarrow x^*$ attracting
- $|f'(x^*)| > 1 \Rightarrow x^*$ repelling
- $|f'| = 1 \Rightarrow$ cannot tell (unconclusive)

Proof: of (i) $|f'(x^*)| < 1 \Rightarrow x^*$ attracting.

Let $|f'(x^*)| = \nu < 1$.
Let $\nu < K < 1$.
f' continuous $\Rightarrow$ ($\exists$ neighborhood s.t. $\nu < K$)
$\exists \delta > 0$ s.t. $|f'(x)| < K$, $\forall x \in I := [x^*-\delta, x^*+\delta]$
Let $x \in I \setminus \{x^*\}$

$$\frac{f(x) - x^*}{x - x^*} = \frac{f(x) - f(x^*)}{x - x^*} = f'(c) \quad \text{for } c \in [x, x^*] \text{ or } [x^*, x]$$
by Mean Value Thm

$|f'(c)| < K$ because $x \in I$
$\Rightarrow |f(x) - x^*| < K|x - x^*|$.

Consider
$$\frac{f(f(x)) - f(f(x^*))}{f(x) - f(x^*)} \underset{\text{by MVT}}{=} f'(\tilde{c}) \quad \text{with } \tilde{c} \in [f(x), f(x^*)] \subseteq I$$
because $|f(x) - x^*|$ less than $K|x - x^*|$

$$= \frac{f^2(x) - x^*}{f(x) - x^*} \Rightarrow |f^2(x) - x^*| = |f'(\tilde{c})||f(x) - x^*| < K\underbrace{|f(x) - x^*|}_{K|x-x^*|}$$

$$\Rightarrow |f^2(x) - x^*| < K^2|x - x^*|$$

$\leadsto |f^n(x) - x^*| < K^n|x - x^*|$ by induction

We say that a period n point x^* is attracting (repelling) if x^* is an attracting (repelling) point of f^n.

$|(f^n)'(x^*)| < 1 \Rightarrow x^*$ is attracting.

$|(f^n)'(x^*)| > 1 \Rightarrow x^*$ is repelling.

Note: $(f^n)'(x^*) = (f^n)'(x_j) \quad \forall\, x_j = f^j(x^*)$

$\Rightarrow$ we can determine this from any point on the orbit of x^*.

Fixed points and their stability

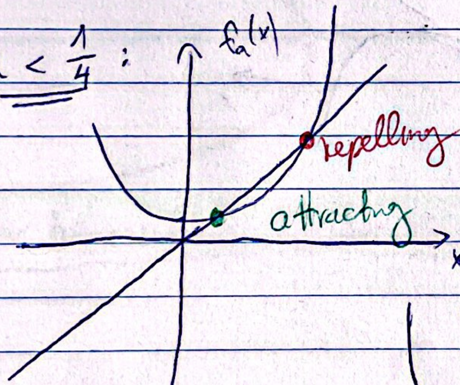
Bifurcations of fixed points

☆ $f_a(x) = x^2 + a$, $a \in \mathbb{R}$ parameter

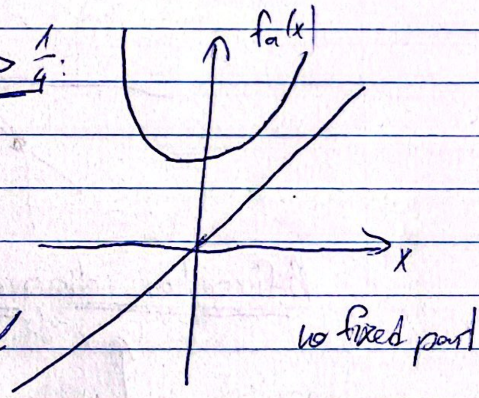
fixed points

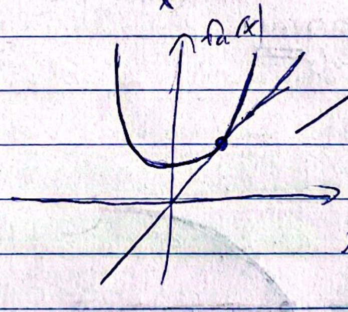
$$f_a(x) = x \iff x^2 - x + a = 0$$
$$\iff x = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4a}$$

$a < \frac{1}{4}$:

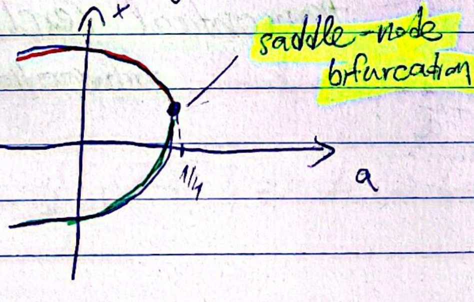


$a > \frac{1}{4}$:





bifurcation diagram:



☆ $f_a(x) = ax(1-x)$, $a > 0$ – logistic map

fixed points: $ax(1-x) = x$ $\Leftrightarrow$

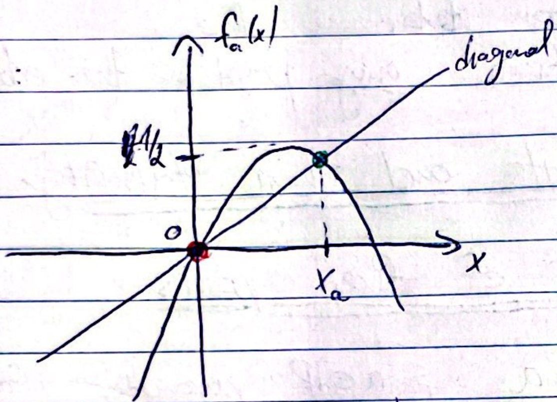
$\underline{x_0 = 0}$ $\vee$ $a(1-x) = 1$

$1 - x = \frac{1}{a}$

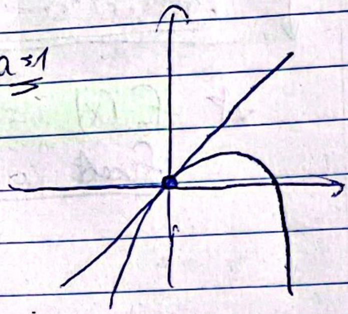
$x = 1 - \frac{1}{a}$

$\underline{x_a = \frac{a-1}{a}}$

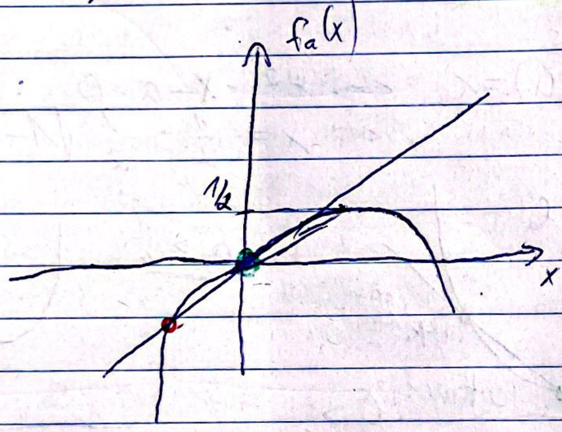
$\underline{a > 1}$:



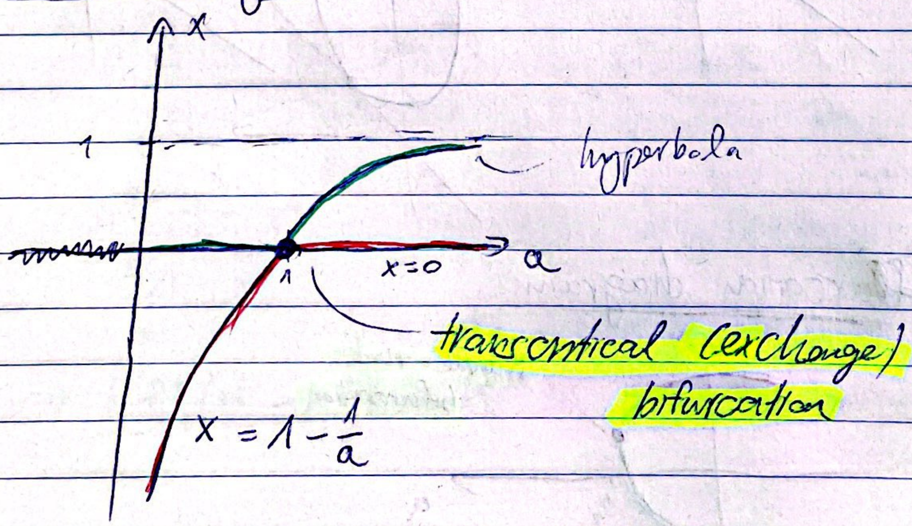
$\underline{a = 1}$



$\underline{0 < a < 1}$:



Bifurcation diagram



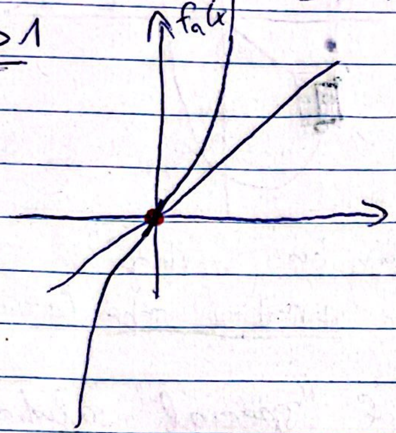
☆ $f_a(x) = ax + x^3 = x(a+x^2)$

fixed points

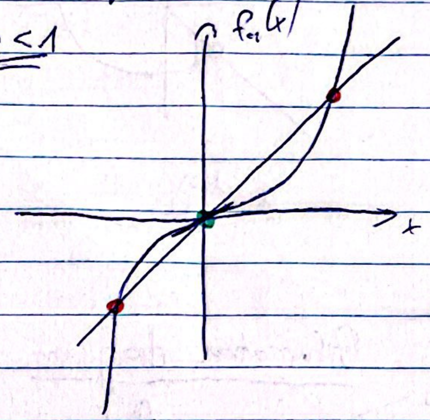
$f_a(x) = x \iff ax + x^3 = x$

$\iff x = 0 \lor x = \pm\sqrt{1-a}$

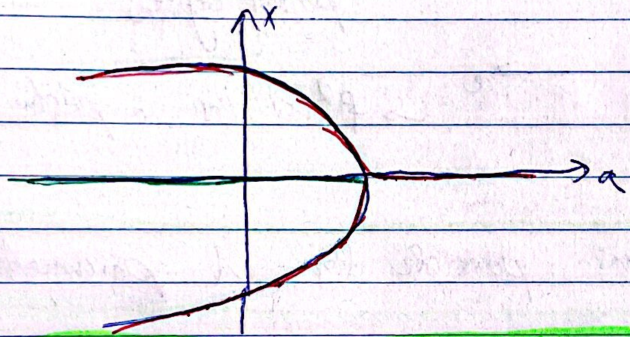
a>1



a<1



Bifurcation diagram:



(supercritical pitchfork bifurcation)

For bifurcations for discrete case: $x_{n+1} = f(x_n)$

i) $f_a(x) = x$ – fixed point

ii) $|f_a'(x)| = 1$ – not hyperbolic

+ more conditions based on type of bifurcation

☆ period doubling bifurcation

previously: we had $f_a'(x) = 1$ at bifurcation

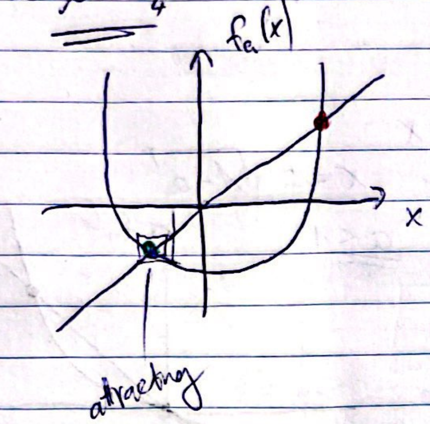
now: consider $f_a'(x) = -1$ at bifurcation

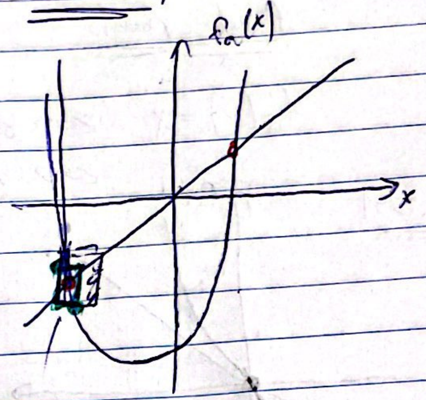
$f_a(x) = x^2 + a$, fixed points $x = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4a}$

$f_a'(x) = 2x$

$f_a'\left(\frac{1}{2} - \frac{1}{2}\sqrt{1-4a}\right) = 1 - \sqrt{1-4a} = -1$ for $\underline{a = -\frac{3}{4}}$

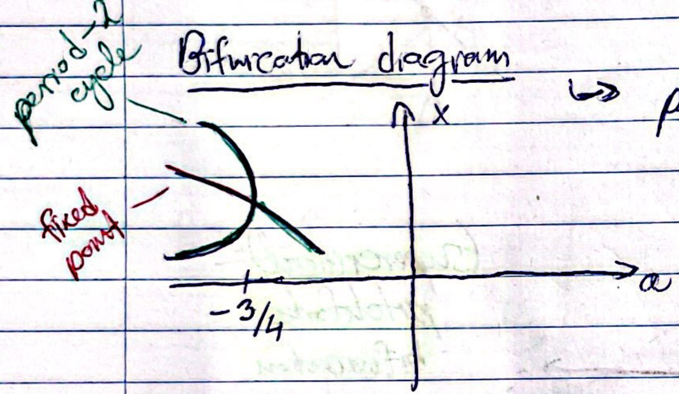
$a \gtrsim -\frac{3}{4}$ $\qquad\qquad$ $a \lesssim -\frac{3}{4}$





repelling but nearing a period-2 orbit (attracting)

Bifurcation diagram $\rightsquigarrow$ plot of "special" solutions
- fixed points
- period cycles

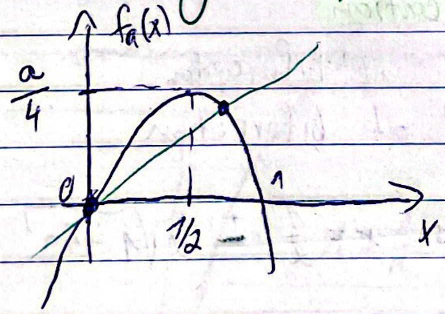


$\rightsquigarrow f^2(x)$ has a pitchfork

$\boxed{\text{All bifurcations possible for 1 parameter}}$

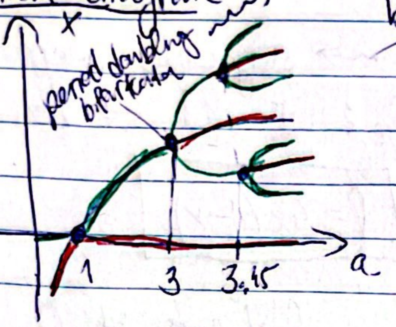
Chaos

$f_a(x) = ax(1-x)$, $4 \geq a > 0$
logistic map $\qquad x \in [0,1] \rightsquigarrow f_a(x) \in [0,1]$



$f_a(x) = x \iff x = 0 \ \vee \ x = 1 - \frac{1}{a}$

Bifurcation diagram → looks like a pitchfork but both branches correspond to same period cycle rather than a fixed point



exchange / transcritical bifurcation at $(a,x) = (1,0)$

$f_a'(x) = a(1-2x)$

$f_a'(0) = a$

$f_a'(1-\frac{1}{a}) = 2-a$

$\Rightarrow$ attracting for $a \in (1,3)$

period-2 points

$$f_a^2(x) = f_a(f_a(x)) = \underbrace{a(ax(1-x))(1-ax(1-x))}_{\text{polynomial of order 4 eq.}} = x$$

$\Leftrightarrow a^2x(1-x)(1-ax(1-x)) - x = 0$

$(a^2x - a^2x^2)(1-ax+ax^2) - x = 0$

~~$a^2x - a^2x^2 - a^3x^2 +$~~

? 0's of 4th polyn.?

⇝ not too hard because 0 and $1-\frac{1}{a}$ are roots $\Rightarrow$ factor out

$\Rightarrow$ obtain polynomial of degree 2

$\Rightarrow$ find roots

$\Rightarrow$

$$x_\pm = \frac{1}{2a}\left(a+1 \pm \sqrt{a^2-2a-3}\right)$$

↳ real when $\sqrt{\bigcirc} \geq 0 \Leftrightarrow \boxed{a \geq 3}$

$\Rightarrow$ period-2 cycle appears at $a = 3$ and grows

$f_a(x_\pm) = x_\mp$

Stability:

$$(f_a^2)'(x_\pm) = f_a'(x_+)\, f_a'(x_-) = 4 + 2a - a^2 = \begin{cases} +1 \text{ for } a = -1, 3 & \text{stable} \\ -1 \text{ for } a = 1 \mp \sqrt{6} & \end{cases}$$

becomes unstable and new period-doubling bifurcation on the branches (reoccurs) $\Rightarrow$ period doubling cascade

$a = 1+\sqrt{6}$

3.45

Consider: convergent sequence

$$\boxed{\frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}} \to 4.662\ldots}$$

Feigenbaum const.

$a_\infty \approx 3.5699\ldots$

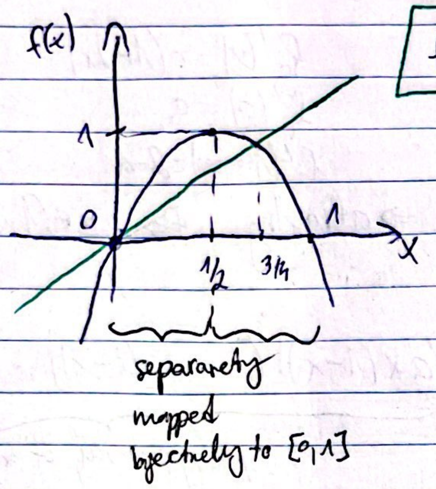
$\Rightarrow$ sequence of a_n's = a's where doubling bifurcation occurs

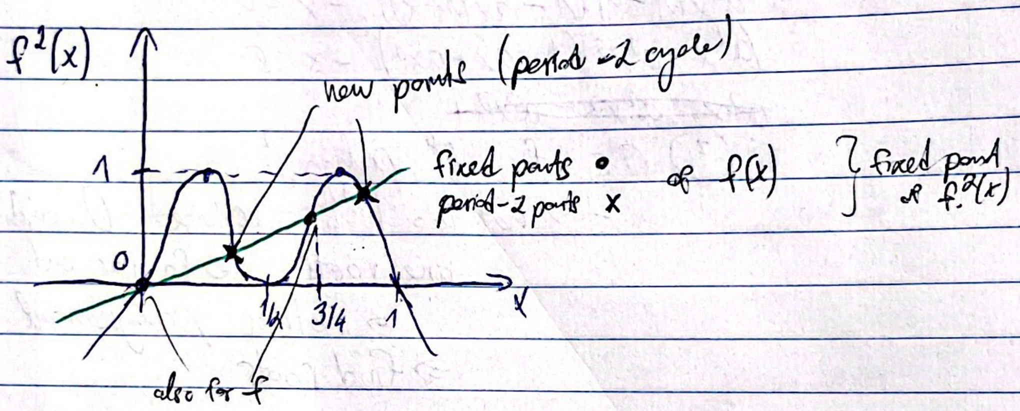
accurs on any map which has a single maximum and gives same Feigenbaum const

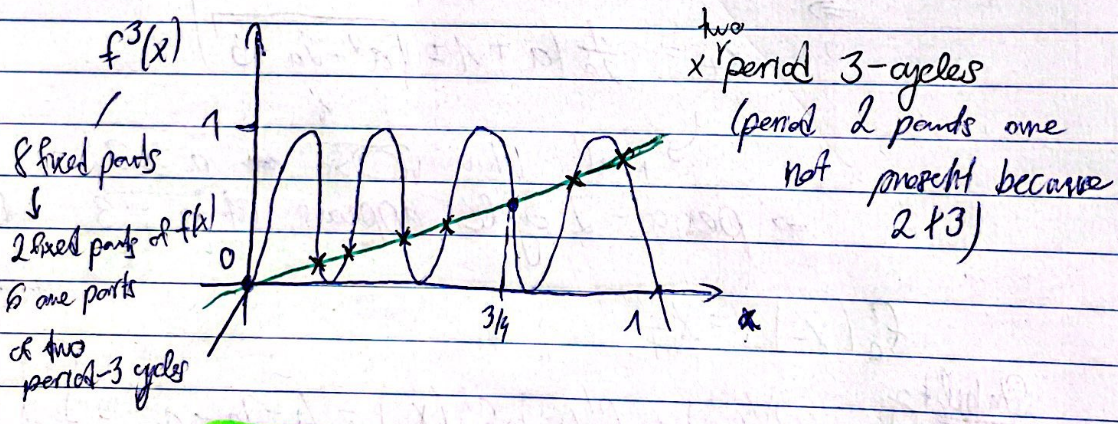
- consider now $\underline{\underline{a=4}}$ $\rightsquigarrow$ surjective mapping of $f = f_4 : [0,1] \mapsto [0,1]$

$$\boxed{f = 4x(1-x)}$$

fixed points at 0 and $3/4$







$\Rightarrow$ $\boxed{\text{$f^n$ has 2^n-many fixed points and 2^n subintervals of $[0,1]$ are mapped bijectively to $[0,1]$. Length of these subintervals $\xrightarrow[n\to\infty]{} 0$ (become shorter)}}$

$\Rightarrow$ period points are dense in $[0,1]$ because each subinterval contains periodic points.

system is often not chaotic everywhere but only on some neighbourhood.

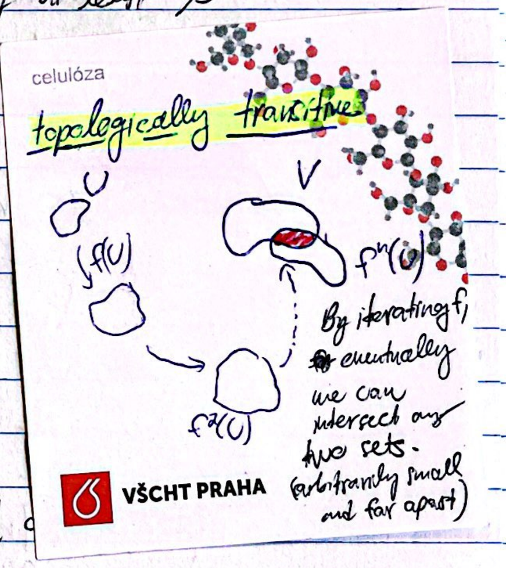
Def: Let ϕ_t be the flow of a continuous time system in $\mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{R}^n$ defining a discrete time system
$$x_{k+1} = f(x_k), \quad k = 0,1,2,\dots$$
The dyn. sys. is chaotic on a compact invariant interval $\Lambda \subseteq \mathbb{R}^n$ if:

1) periodic points are dense in Λ
2) ϕ_t resp. f is topologically transitive
   = for any open sets U and V, $\exists t>0$ $(n>0)$
   s.t. $\phi_t(U) \cap V \neq \emptyset$ $\quad (f^n(U) \cap V \neq \emptyset)$

$1)+2) \Rightarrow$ on intervals

3) ϕ_t / f has sensitive dependence on initial conditions,
   global = $\exists \beta > 0$, s.t. for any $x_0 \in \Lambda$ and any neighbourhood U of x_0, $\exists y_0 \in U$ and $t>0$ $(n>0)$
   s.t. $\|\phi_t(x_0) - \phi_t(y_0)\| > \beta$ = sensitivity parameter
   $(\|f^n(x_0) - f^n(y_0)\| > \beta)$

points which start off arbitrarily close together will eventually be separated by at least β

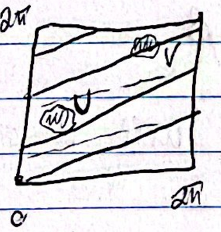


$(\theta_1, \theta_2) \in S^1 \times S^1 = \mathbb{T}^2$, $\quad S^1 = \mathbb{R}/(2\pi\mathbb{Z})$

$$\begin{cases} \frac{d}{dt}\theta_1 = \omega_1 \\ \frac{d}{dt}\theta_2 = \omega_2 \end{cases}, \quad \frac{\omega_1}{\omega_2} \in \mathbb{R}\setminus\mathbb{Q}, \quad \omega_1, \omega_2 \in \mathbb{R}$$

$$\phi_t(\theta_1, \theta_2) = (\theta_1 + \omega_1 t, \theta_2 + \omega_2 t)$$

View torus as a square with identified e



$\frac{\omega_1}{\omega_2}$ irrational $\Rightarrow$ solution fills in densly the entire square
$\Downarrow$
no periodic points

- X • no periodic points ($\emptyset$ is dense ?) X
- V • topological transitivity ✓
- X • no sensitive dependence on initial conditions X
  ↳ points staying close, stay close ↗↗ run parallel

NOT chaotic ⇐

A discrete time system $\nearrow$ const. $r \Rightarrow$ points stay on circles

$f(r,\theta) = (r, \; 2\pi\,\Omega(r) + \theta)$

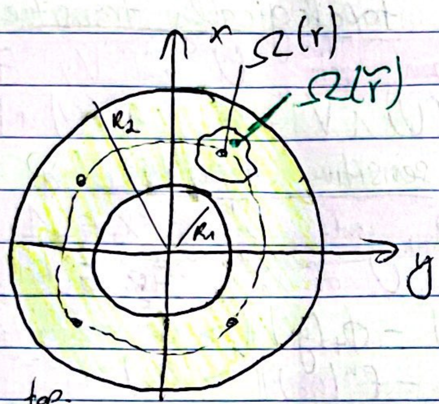
where $\Omega : [0,\infty) \to [0,\infty)$, C^∞ function

$\downarrow$ map on $\mathbb{R}^2$

$\frac{\partial \Omega}{\partial r}(r) \neq 0$

but consider subregion: $(r,\theta) = [R_1, R_2] \times \mathbb{R}/(2\pi\mathbb{Z}) = Ann$

annulus $0 < R_1 < R_2$



✓ periodic points, p/q ✓

$\Leftrightarrow \Omega(r) \in \mathbb{Q}$,

Ann $f^q(r,\theta) = (r, 2\pi q \cdot \frac{p}{q} + \theta q) =$

$= (r, 2\pi p + \theta q) =$

$= (r, \theta)$

$\hookrightarrow$ periodic points are dense because $\mathbb{Q}$ is dense in $\mathbb{R}$

$\hookrightarrow$ any open set $U \in Ann$ contains (r,θ) for which $\Omega(r)$ is rational

X. top. transitivity X

- circles are invariant (r = const. for motion)

$\hookrightarrow$ if we begin on one side of a circle, we cannot get to the other side

✓. sensitivity ✓

- prove $\Omega(r) - \Omega(\tilde r) \in \mathbb{R}\setminus\mathbb{Q}$

$\Rightarrow$ r and $\tilde r$ points don't come closer again

- let $\beta = R_1$ (or even $2R_1$)

Choose $(r,\tilde\theta) \in U \subseteq Ann$ and $(\tilde r,\theta) \in U$,

s.t. $\Omega(r) - \Omega(\tilde r) \in \mathbb{R}\setminus\mathbb{Q}$

$\Rightarrow \exists n > 0$, $f^n(\tilde r, \theta)$ has distance to $f^n(r,\tilde\theta)$ greater than β

eventually r and $\tilde r$ points will be on opposite sides of the circle

☆ Shift map (typical example of chaotic system)

State space: $\Sigma := \{ S : \mathbb{Z}_{\geq 0} \to \{0,1\} \}$

set of sequences of 0's and 1's

$= \{ (s_0, s_1, s_2, \dots) \mid s_k \in \{0,1\},\ k = 0,1,2,\dots \}$

$\boxed{\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}}$

define a metric:

$$d : \Sigma \times \Sigma \to \mathbb{R}$$
$$(s,t) \mapsto \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} \leq \sum_{i=0}^{\infty} \frac{1}{2^i} = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = \frac{1}{1-\frac{1}{2}} = 2 \ ✓$$

geo series

conditions to be a metric:
- ☑ symmetric
- ☑ Δ inequality
- ☑ positive, definite (and 0 only when $s=t$)

Shift map: $\sigma : \Sigma \to \Sigma$

$$s = (s_0, s_1, s_2, \dots) \mapsto (s_1, s_2, s_3, \dots)$$

Lemma: $s,t \in \Sigma$

i) $s_j = t_j$ for $j = 0, \dots, n \ \to \ d(s,t) \leq \frac{1}{2^n}$

ii) $d(s,t) < \frac{1}{2^n} \ \Rightarrow \ s_j = t_j \ \forall j = 0, \dots, n$

Proof: i) $s_j = t_j$ for $j = 0, \dots, n$

$$d(s,t) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i} = \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i} = \sum_{i=0}^{\infty} \frac{|s_{i+n+1} - t_{i+n+1}|}{2^{i+n+1}} =$$

$$= \frac{1}{2^{n+1}} \sum_{i=0}^{\infty} \frac{\overbrace{|s_{i+n+1} - t_{i+n+1}|}^{\leq 1}}{2^i} \leq \frac{1}{2^{n+1}} \sum_{i=0}^{\infty} \frac{1}{2^i} =$$

$$= \frac{1}{2^{n+1}} \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^i = \frac{1}{2^{n+1}} \frac{1}{1-\frac{1}{2}} = \frac{1}{2^{n+1}} \frac{1}{\frac{1}{2}} = \frac{2}{2^{n+1}} = \frac{1}{2^n}$$

$\Rightarrow d(s,t) \leq \frac{1}{2^n}$ □

ii) Similarly

Proposition: $\sigma : \Sigma \to \Sigma$ continuous

Proof: Let $s = (s_0, s_1, s_2, \dots) \in \Sigma$ and $\varepsilon > 0$.

Choose $n \in \mathbb{Z}_{\geq 0}$ s.t. $\frac{1}{2^n} < \varepsilon$. Set $\delta = \frac{1}{2^{n+1}}$

Suppose $t \in \Sigma$ with $d(s,t) < \delta$.

Show: $d(\sigma(s), \sigma(t)) < \varepsilon$.

Lemma, iii)

$d(s,t) < \delta \iff d(s,t) < \frac{1}{2^{n+1}} \overset{\downarrow}{\Longrightarrow} t = (s_0, s_1, \dots, s_{n+1}, t_{n+2}, \dots)$

$\Rightarrow \sigma(t) = (s_1, \dots, s_{n+1}, t_{n+2}, \dots)$

$\sigma(s) = (s_1, \dots, s_{n+1}, s_{n+2}, \dots)$

$d(\sigma(s), \sigma(t)) \leq \frac{1}{2^n} < \varepsilon$ by lemma i) because the first n entries agree. □

---

① periodic points of $\sigma: \Sigma \to \Sigma$ are dense in Σ.

Let $t \in \Sigma$ and $\varepsilon > 0$.

Show: $\exists s \in \Sigma$ periodic with $d(s,t) < \varepsilon$.

Choose $n \in \mathbb{Z}_{>0}$ s.t. $\frac{1}{2^n} < \varepsilon$.

Set $s := (\underbrace{t_0, t_1, \dots, t_n}_{n+1 \text{ entries}}, t_0, t_1, \dots, t_n, \dots)$

periodic sequence

repeat blocks of length $n+1$

By lemma $d(s,t) \leq \frac{1}{2^n} < \varepsilon$ because s and t agree in first $n+1$ entries

② σ is topologically transitive.

Show: $\exists s^* \in \Sigma$ with $\sigma^k(s^*)$, $k = 0, 1, 2, \dots$ dense in Σ.

↳ $\{\sigma^k(s^*)\}$ = sequence of sequences

($\Rightarrow$ topological transitivity)

Set $s^* := (\underbrace{0, 1}_{\text{all blocks of length 1}}, \underbrace{0, 0, 0, 1, 1, 0, 1, 1}_{\text{all blocks of length 2}}, \underbrace{0, 0, 0, 0, 0, 1, \dots}_{\text{all possible blocks of length 3}})$

$\dots$ 4, 5, ...

↳ dense in Σ:

- For any ε, we need to find $\sigma^k(s^*)$ in ε-neighborhood of a given point.
- Take n s.t. $\frac{1}{2^n} < \varepsilon$, then take $\sigma^k(s^*)$ to reach the part with all possible blocks of length n. One of them will match the starting sequence of the given point.
- Use lemma

③ σ has sensitivity dependence on initial conditions.

$\beta := 1$. Let $s \in \Sigma$, $\varepsilon > 0$.

β for full state space!

Show: $\exists t \in \Sigma$ with distance $d(s,t) < \varepsilon$

$\exists k \in \mathbb{Z}_{\geq 0}$ s.t. $d(\sigma^k(s), \sigma^k(t)) > \beta$.

Choose $n \in \mathbb{Z}_{\geq 0}$ s.t. $\frac{1}{2^n} < \varepsilon$, $t = (s_0, s_1, s_2, \dots, s_n, \hat{s}_{n+1}, \hat{s}_{n+2}, \dots)$

with $\hat{s}_j = \begin{cases} 1 & \text{if } s_j = 0 \\ 0 & \text{if } s_j = 1 \end{cases}$ (exactly opposite to get them be far apart)

Set $k = n+1 \Rightarrow \underset{\uparrow}{d(\sigma^k(s), \sigma^k(t))} = \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1-\frac{1}{2}} = 2 > 1 = \beta$

all agreeing distances removed, remaining all disagreeing distances

∎

⇒ **Shift map is chaotic.**

$$\boxed{\text{Thm: } \sigma : \Sigma \to \Sigma \text{ defines a chaotic dynamical system.}}$$

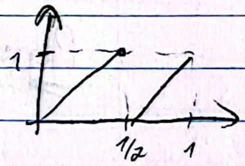
Proof above.

also true for semi-conjugacy

- If ~~[illegible]~~ a system is topologically conjugate to the shift map, the system is ~~top~~ also chaotic — chaos remains under topological conjugacy.

☆ **Doubling map** $d : [0,1] \to [0,1]$

$x \mapsto 2x \mod 1$



$$\boxed{\text{Proposition: } d \text{ is semi-conjugate to } \sigma}$$

↳ h is surjective and each $x \in [0,1]$ has only finitely many pre-images

Proof: $h : \Sigma \to [0,1]$ binary representation

$(x_0, x_1, \dots) \mapsto \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}}$

$9.\overline{9} = 10$

- Show: h is a homeomorphism = bijection and cont. inverse
- Show: $d(h(x_0, x_1, \dots)) = \text{----} = h(\sigma(x_0, x_1, \dots))$

$\left.\begin{matrix}(0,1,1,\dots) \\ (1,0,0,\dots)\end{matrix}\right\} \frac{1}{2} \Rightarrow$ not injective but surjective

only mildly violated injectivity though, ⇝ finitely many images

$d(h(x_0, x_1, \dots)) = 2 \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}} \mod 1 = \sum_{i=0}^{\infty} \frac{x_i}{2^i} \mod 1 = \sum_{i=-1}^{\infty} \frac{x_{i+1}}{2^{i+1}} \mod 1 =$

$= \left(\underbrace{\frac{x_0}{2^0}}_{\substack{x_0 = 0 \vee 1 \\ \Downarrow \\ = 0 \mod 1}} + \underbrace{\sum_{i=0}^{\infty} \frac{x_{i+1}}{2^{i+1}}}_{\leq 1} \right) \mod 1 = \sum_{i=0}^{\infty} \frac{x_{i+1}}{2^{i+1}} = h(\sigma(x_0, x_1, \dots))$

$\left| \sum_{i=0}^{\infty} \frac{x_{i+1}}{2^{i+1}} \right| \leq \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2} \cdot 2 = 1$

Geometric interpretation

- h is an itinerary map. $I_0 = [0, 1/2]$, $I_1 = [1/2, 1]$

for $(x_0, x_1, \dots)$: $d^k(h(x)) = \begin{cases} I_0 & \text{if } x_k = 0 \\ I_1 & \text{if } x_k = 1 \end{cases}$

(Σ, σ) are symbolic dynamics of $([0,1], d)$

Thm: $I, J \subset \mathbb{R}$ intervals and $f: I \to I$, $g: J \to J$, maps C^∞ and semiconjugate by $h: I \to J$, i.e. $g \circ h = h \circ f$.
If $f: I \to I$ chaotic $\Rightarrow$ $g: J \to J$ is chaotic.

Corollary: Doubling map is chaotic.

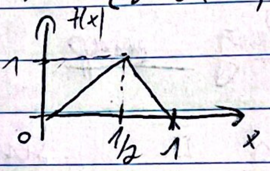
Proof: d is semiconjugate to σ (Thm holds more generally)

☆ Tent map $f: [0,1] \to [0,1]$

$$f(x) := \begin{cases} 2x & , 0 \le x \le 1/2 \\ 2-2x & , 1/2 \le x \le 1 \end{cases}$$

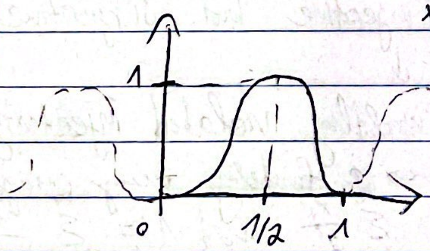
Prop: Tent map is chaotic

Proof: direct ~~in~~ ~~way~~



Prop: Logistic map $\ell: [0,1] \to [0,1]$
$x \mapsto \ell(x) = 4x(1-x)$
is semiconjugate to the tent map $f: [0,1] \to [0,1]$.
$\Rightarrow$ chaotic

Proof: Set $h: [0,1] \to [0,1]$
$x \mapsto h(x) = \frac{1}{2}(1 - \cos(2\pi x))$



surjective ✓
each image point at most 2 preimages ✓

periodicity and symmetry of cos

$$h(f(x)) = \begin{cases} \frac{1}{2}(1-\cos(2\pi \cdot 2x)), & 0 \le x \le 1/2 \\ \frac{1}{2}(1-\cos(2\pi \cdot (2-2x)), & 1/2 \le x \le 1 \end{cases} = \frac{1}{2}(1-\cos(4\pi x)) =$$

$$= \frac{1}{2}(2 - 2\cos^2(2\pi x)) = 4\left(\frac{1}{4} - \frac{1}{4}\cos^2(2\pi x)\right) =$$

$$= 4\left(\frac{1}{2} - \frac{1}{2}\cos(2\pi x)\right)\left(\frac{1}{2} + \frac{1}{2}\cos(2\pi x)\right) = 4\,h(x)\left(1 - \frac{1}{2} + \frac{1}{2}\cos(2\pi x)\right)$$

$$= 4\,h(x)\left(1 - \left(\frac{1}{2} - \frac{1}{2}\cos(2\pi x)\right)\right) = 4h(x)(1-h(x))$$

$$f(h(x)) = 4(h(x))(1-h(x))$$